Some further refinements and extensions of the Hermite–Hadamard and Jensen inequalities in several variables

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\textbf{ABSTRACT}

The main object of this paper is to give several refinements and extensions of the Hermite–Hadamard and Jensen inequalities in \( n \) variables. Relevant connections of the results presented here and the various inequalities derived in earlier investigations are also indicated.

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1. Introduction

Let
\[
x = (x_1, \ldots, x_n) \quad \text{and} \quad N_0 := \mathbb{N} \cup \{0\} \quad (\mathbb{N} := \{1, 2, 3, \ldots\}).
\]

Also let \( I \) be a convex subset of an arbitrary real linear space \( X \). A function \( f : I \to \mathbb{R} \) is called convex if, for every two elements \( a, b \in I \), the following inequality holds true:
\[
f \left( \frac{a + b}{2} \right) \leq \frac{f(a) + f(b)}{2}.
\]

We begin by recalling the following known results.

\textbf{Theorem 1.} (see [1,2]) For every convex function \( f \), the Jensen inequality:
\[
f \left( \frac{\sum_{i=1}^{n} x_i}{n} \right) \leq \frac{1}{n} \sum_{i=1}^{n} f(x_i)
\]

and the weighted Jensen inequality:
\[
f \left( \frac{\sum_{i=1}^{n} p_i x_i}{p_n} \right) \leq \frac{1}{p_n} \sum_{i=1}^{n} p_i f(x_i)
\]

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