A method to refine the discrete Jensen’s inequality for convex and mid-convex functions

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A general method is developed to refine the discrete Jensen’s inequality in the convex and mid-convex cases. A number of refinements of the discrete Jensen’s inequality can be obtained by using the method. The results generalize well known inequalities and give also a new treatment of them. The results are applied to some special situations.

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1. Introduction and the main result

The fundamental discrete Jensen’s inequalities for convex and mid-convex functions say that

**Theorem A** (See [1]). Let $C$ be a convex subset of a real vector space $X$, let $x_i \in C$, and let $p_i \geq 0$ ($i = 1, \ldots, n$) with $\sum_{i=1}^{n} p_i = 1$.

(a) If $f : C \rightarrow \mathbb{R}$ is a convex function, then

$$f \left( \sum_{i=1}^{n} p_i x_i \right) \leq \sum_{i=1}^{n} p_i f (x_i) .$$

(1)

(b) If $f : C \rightarrow \mathbb{R}$ is a mid-convex function, and $p_i$ is rational ($i = 1, \ldots, n$), then (1) also holds.

Here the function $f : C \rightarrow \mathbb{R}$ is called convex if

$$f (\beta x + (1 - \beta) y) \leq \beta f (x) + (1 - \beta) f (y), \quad x, y \in C, \quad 0 \leq \beta \leq 1,$$

and mid-convex if

$$f \left( \frac{x + y}{2} \right) \leq \frac{1}{2} f (x) + \frac{1}{2} f (y), \quad x, y \in C.$$

In the paper [2] we give a refinement of Theorem A, which generalizes and unifies some previous results (see [3,4]). The following conditions are used in [2] and they will be essential in the sequel too.

(H1) Let $V$ be a real vector space, let $C$ be a convex subset of $V$, and let $x_1, \ldots, x_n \in C$, where $n \geq 1$ is a fixed integer.

(H2) Let $p_1, \ldots, p_n > 0$ such that $\sum_{j=1}^{n} p_j = 1$.

(H3) Let the function $f : C \rightarrow \mathbb{R}$ be convex.

(H4) Let the function $f : C \rightarrow \mathbb{R}$ be mid-convex, and let $p_1, \ldots, p_n$ be rational.

Now we recall the central result of [2].

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