International Journal of Non-Linear Mechanics 46 (2011) 742–744

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

On the ascent and descent times of a projectile in a resistant medium

F.J. Rooney [a,*], S.K. Eberhard [b]

[a] Bishop O'Dowd, 9500 Stearns Avenue, Oakland, CA 94605, USA
[b] Gonville & Caius College, Trinity Street, Cambridge CB2-1TA, UK

ARTICLE INFO

Article history:
Received 15 October 2010
Accepted 22 February 2011
Available online 2 March 2011

Keywords:
Projectile motion
Ascent time
Descent time
Resistance

ABSTRACT

The motion of a projectile in a uniform gravitational field loses its symmetry when a resisting force is present, essentially because Newton's law loses its reversibility. Allowing the magnitude of the resistance to depend arbitrarily on speed, the motion is governed by two coupled first-order non-linear differential equations. Though intractable to solve explicitly, these equations can be made to yield much qualitative information about the trajectory. The present paper focuses on the ascent and descent times of the projectile, providing a proof that the ascent and descent times are bounded, above and below respectively, by the corresponding (equal) times of a projectile reaching the same height without resistance. Additionally, the difference is shown to increase with the velocity of projection. Direct corollaries are two well known observed features of the motion: The time of ascent is always less than the time of descent, and the difference increases with the velocity of projection. For the bounds themselves, the resistance is only assumed to be positive, but to show that the difference increases with the velocity of projection requires the additional assumption that the resistance increases at least linearly with speed.

1. Introduction

The apparent asymmetry of a projectile's path in air subject to the Earth's gravity is a well-known effect of air resistance that runs up against the symmetry of the theoretical motion in a vacuum given in high-school physics courses. Historically, one of the first asymmetries to be observed and studied is the discrepancy between the ascent and descent times. In response to Galileo's parabolic model of projectile motion, Marin Mersenne conducted several experiments with a crossbow, observing in 1644 that "the time of ascent is always less than the time of descent, the difference increasing with the velocity of projection" [5]. Though these properties, referred to in this paper as Properties 1 and 2, may be familiar and even intuitive, the presence of a resistance term in the equation of motion sufficiently complicates the dynamics that a general analytic proof has not yet been seen.

Several different forms of the resistance (also called "drag") as a function of speed have been studied. Assuming the resistance is linear in speed is often convenient for analytic purposes because it gives rise to linear, uncoupled differential equations from which a simple closed-form expression for the position as a function of time is obtained. In this case, Properties 1 and 2 can be proved at the level of elementary calculus. See, for example, Groetsch [2]. The accuracy, however, of the so-called "linear model" is suspect. Indeed, if one assumes that the only parameters affecting the drag force are speed V, air density ρ, and cross-sectional area A, then dimensional analysis suggests the relationship

$$F_d = \tfrac{1}{2} C_d \rho A V^2,$$

where C_d is a dimensionless constant called the drag coefficient, the factor of 1/2 being conventional. This is called the "quadratic model", and it is considered distinctly superior for everyday circumstances. Hayen [3] analysed this model and proved several qualitative features, including Property 1 for vertical flight. Though his results can probably be used to prove Property 1 for general launch angles he does not appear to pursue this, nor does he mention Property 2. In general, to good approximation, the dependence of the drag on speed is determined by a dimensionless quantity called the *Reynolds number*, defined in this case by

$$Re = \frac{V\sqrt{A}}{\nu},$$

where ν is the kinematic viscosity of the fluid (about $1.5 \times 10^{-5}\ \mathrm{m^2/s}$ for dry air). If $Re \gg 1$, such as in air or water at everyday speeds, inertial forces are dominant and the quadratic model is appropriate, while if $Re \ll 1$, such as in syrup or at small speeds, viscous forces are dominant and the linear model is

* Corresponding author. Tel.: +1 510 577 9100; fax: +1 510 638 3259.
E-mail addresses: frooney@ix.netcom.com (F.J. Rooney), se288@cam.ac.uk (S.K. Eberhard).

0020-7462/$ - see front matter © 2011 Published by Elsevier Ltd.
doi:10.1016/j.ijnonlinmec.2011.02.007