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Generating functions for volume-preserving transformations

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ABSTRACT

A general implicit solution for determining volume-preserving transformations in the n-dimensional Euclidean space is obtained in terms of a set of $2n$ generating functions in mixed coordinates. For $n=2$, the proposed representation corresponds to the classical definition of a potential stream function in a canonical transformation. For $n=3$, the given solution defines a more general class of isochoric transformations, when compared to existing methods based on multiple potentials. Illustrative examples are discussed both in rectangular and in cylindrical coordinates for applications in mechanical problems of incompressible continua. Solving exactly the incompressibility constraint, the proposed representation method is suitable for determining three-dimensional isochoric perturbations to be used in bifurcation theory. Applications in non-linear elasticity are envisaged for determining the occurrence of complex instability patterns for soft elastic materials.

1. Introduction

Considering a bounded region Ω_0 in the n-dimensional Euclidean space, this work is aimed at defining generating functions for volume-preserving transformations of a set of continuously differentiable functions $u_j = u_j(U_1,U_2,\ldots,U_n) : \Omega_0 \to \Re$, with $j=1,2..,n$. Such an isochoric constraint can be expressed by a non-linear first-order partial differential equation as follows:

$$J(U_1,U_2,\ldots,U_n) = \det \frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(U_1,U_2,\ldots,U_n)} = 1 \tag{1}$$

where J is defined as the Jacobian of the transformation. The cases $n=2,3$ are of particular interests in continuum mechanics, because the functions U_j,u_j can be treated as the material/spatial components of the position vectors $\mathbf{U}$, $\mathbf{u}=\mathbf{u}(\mathbf{U})$ in the reference/actual configuration, respectively. In such a case, the Jacobian defined in Eq. (1) corresponds to the determinant of the deformation tensor $\mathbf{F} = \mathrm{Grad}\,\mathbf{u} = \partial\mathbf{u}/\partial\mathbf{U}$, so that the functions u_j determine the deformation fields for an incompressible material. For $n=2$, the solution of Eq. (1) corresponds to an area-preserving transformation, as reported by Bateman [1], who ascribed its first formulation to Gauss. Rooney and Carroll [7] realized that such a solution could be expressed by an implicit representation through the definition of a stream function. Using this change of notation, the governing equations have the structure of Hamilton's canonical equations with one degree of freedom, therefore such a stream function can be regarded as a generating function for a canonical transform of planar coordinates. The extension of this solution to $n \geq 3$ was considered by Carroll [4], who proposed an implicit representation by the means of $(n-1)$ potential functions, restricted by a set of $(n-1)$ admissibility conditions. Another implicit solution was later proposed by Knops [6], transforming the problem to a linear first-order non-homogeneous differential equation by using prescribed cofactors in the expanded expression for the Jacobian, recovering Carroll's expression for $n=3$. Although representing complete solutions of the differential problem given by Eq. (1), both methods are given in implicit form and their application might be difficult for seeking explicit solutions with given boundary conditions imposed by the mechanical problem under consideration.

This work is organized as follows. In Section 2, the existing description of volume-preserving transformation using coupled potential functions is analyzed, underlying its limitations for continuum mechanics applications. In Section 3, the definition of generating functions for volume-preserving transformation is given for a general n-dimensional problem. The three-dimensional case is particularly examined, highlighting possible applications for stability problems in non-linear elasticity. The results are finally summarized in Section 4.

2. Limitations of existing solutions

In this paragraph, the solution for a generic isochoric deformation presented by Carroll [4] is analyzed. Choosing $n=3$ for the sake of simplicity, the volume-preserving transformation is given in terms of two potential functions $\Phi(X,y,z)$ and $\psi(X,Y,z)$, referring to different mixed coordinate systems. The general solution takes the following implicit form:

$$x = \frac{\partial \Phi(X,y,z)}{\partial y} \tag{2}$$

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