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Dynamics of the nearly parametric pendulum

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Dynamically stable periodic rotations of a driven pendulum provide a unique mechanism for generating a uniform rotation from bounded excitations. This paper studies the effects of a small ellipticity of the driving, perturbing the classical parametric pendulum. The first finding is that the region in the parameter plane of amplitude and frequency of excitation where rotations are possible increases with the ellipticity. Second, the resonance tongues, which are the most characteristic feature of the classical bifurcation scenario of a parametrically driven pendulum, merge into a single region of instability.

1. Introduction

The driven pendulum is a generic model used for studying non-linear dynamics in mechanics [1] and beyond [2–6]. Its geometric non-linearity can be modeled reliably (in contrast to other non-linear effects such as friction), and mechanical pendula are amenable to experimental investigations. The dynamical properties of the classical parametrically driven pendulum, such as resonances, escape from a potential well, symmetry-breaking, and periodic and chaotic attractors, have been explored in detail experimentally [1,7–11] and theoretically [12–27].

This paper investigates what happens to the well-studied bifurcation scenarios of the parametrically excited pendulum if the driving of the pivot of the pendulum follows a narrow upright ellipse; see Fig. 1. One motivation for studying elliptic excitation is that only the elliptic component of an arbitrarily shaped periodic excitation has an effect on a rotating pendulum for large excitation frequencies; see Section 2 for an explanation. Moreover, elliptic excitation is typical if the pendulum base is floating on water waves: a freely floating body moves along an ellipse. This effect is similar to the elliptic motion of an off-center surface point of a plate excited by a circular traveling bending wave (a principle that is exploited in rotary ultrasonic motors [28,29]).

We say that the pendulum *rotates* if the long-time average of the angular velocity is non-zero. Stable periodic rotations occur naturally over a large range of excitation parameters in the parametrically driven pendulum [12]. Thus, a rotating pendulum provides a unique mechanism for generating a uniformly one-directional rotation from a bounded excitation. This is a potential physical principle for harnessing the energy of vibrations, which are not necessarily purely in the vertical direction. The other motivation for focusing on rotating attractors is that the rotating pendulum is ideal for developing and testing non-invasive bifurcation and chaos control methods [30,31] in a real experiment: periodic rotations are reliably controllable by superimposing feedback control onto the excitation without changing the shape of the excitation. This is not true in general for small-amplitude oscillations around the hanging-down position [32].

The dimensionless equation of motion for the elliptically excited pendulum is

$$\ddot{\theta}+\gamma\dot{\theta}+(1+p\cos(\omega t))\sin\theta+ep\sin(\omega t)\cos\theta=0 \tag{1}$$

where γ is the dimensionless viscous damping coefficient, p is the scaled excitation amplitude, ω the rescaled excitation frequency, and e is the ratio between the horizontal and the vertical diameter of the upright ellipse traced out by the pivot during each period (see Fig. 1). The classical parametrically excited pendulum corresponds to the setting $e=0$.

The two main effects of a small non-zero ellipticity e of the excitation are:

1. The classical resonance tongues for the 1:2 and the 1:1 resonance of the parametrically excited pendulum [12] merge into a single region of instability of the small-amplitude period-one oscillation around the hanging-down position of the pendulum.
2. If the ellipticity e is non-zero the pendulum is no longer symmetric with respect to reflection $\theta\mapsto-\theta$, which causes a preference for rotations that have the same direction as the motion of the pivot. Effectively, the range of possible excitation frequencies and amplitudes where rotations are supported increases for increasing ellipticity. The preferred direction of rotation has the same sense as the motion of the base around the ellipse because this rotation picks up energy from the additional excitation into the horizontal direction.

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