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Physically-based Dirac's delta functions in the static analysis of multi-cracked Euler–Bernoulli and Timoshenko beams

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A B S T R A C T

Dirac's delta functions enable simple and effective representations of point loads and singularities in a variety of structural problems, leading very often to elegant and otherwise unworkable closed-form solutions. This is the case of cracked beams under static loads, whose theoretical and practical significance has attracted in recent years the interest of many researchers. Nevertheless, analytical formulations currently available for this problem are not completely satisfactory, either in terms of computational efficiency, when the continuity conditions must be enforced with auxiliary equations, or in terms of physical consistency, when the singularities in the beam's flexural rigidity are represented with Dirac's delta functions having a questionable negative sign. These considerations motivate the present study, which offers a novel and physically-based modelling of slender Euler–Bernoulli beams and short Timoshenko beams with any number and severity of cracks, conducing in both cases to exact closed-form solutions. For validation purposes, a standard finite element code is used, along with two nascent deltas (uniform and Gaussian density functions) to describe a smeared increase in the bending flexibility around the abscissa of the crack.

1. Introduction

Structural analysis of multi-cracked beams is of great engineering interest, and has been extensively studied in the last decades. As a matter of fact, presence of cracks may radically change the behaviour of beams and reduce their performances in statics and dynamics, leading to excessive deflections and unexpected failures.

Research in this field has been mainly concentrated on two classes of problems: (i) definition of appropriate linear and non-linear models for representing the effects of cracks under static and dynamic loadings and (ii) detection of position and severity of the damage by using either static or dynamic tests (e.g., Banan and Hjelmstad, 1994; Dimarogonas, 1996; Hjelmstad and Shin, 1997; Chondros et al., 1998). Belonging to the first stream of research, this paper deals with an effective and physically-based linear modelling of multi-cracked beams subjected to static loadings, although the results presented herein can be useful for treating any type of concentrated damage occurring in slender and short beams, e.g. corrosion of steel bars in reinforced concrete members, defects of material and attacks of biotic agents in timber elements, and etcetera. Indeed, the proposed model enables one to analytically represent a local increase in the bending flexibility of the beam, which is actually what all the types of damage mentioned above have in common. According to the classification by Friswell and Penny (2002), the proposed approach falls in the broad category of "discrete spring models", being equivalent to an internal hinge coupled with a linear elastic spring, which is herein assumed to have constant rigidity independently of the loading direction. Although very simple, this "always open" model (Irwin, 1957) proves to be very efficient for static problems (Buda and Caddemi, 2007; Caddemi and Morassi, 2007; Caddemi and Di Paola, 2008); it can be also applied to dynamic problems when the amplitude of vibration is smaller than the static deflection (Chondros et al., 2001), while "breathing in time" models (Kirmsher, 1944) are mandatory when cracks open and close, in so causing more complicated nonlinear phenomena. Extended finite element method (e.g. Belytschko and Black, 1999; Moës et al., 1999) and meshless methods (e.g. Nguyen et al., 2008; Yaw et al., 2009) are very powerful computational strategies in this context, especially when initiation and propagation of cracks are studied in 2D or 3D models of structural members.

Classical analytical approaches are particularly appealing when the global behaviour of frame structures is concerned. The idea of treating multi-cracked beams with equivalent linear springs at the

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