International Journal of Solids and Structures 48 (2011) 843–853

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

# Closed-form solution of a shear deformable, extensional ring in contact between two rigid surfaces

Amir Gasmi [a], Paul F. Joseph [a,*], Timothy B. Rhyne [b], Steven M. Cron [b]

[a] Department of Mechanical Engineering, Clemson University, 102 Fluor Daniel Building, Clemson, SC 29634, USA
[b] Michelin Americas Research and Development Company, 515 Michelin Road Greenville, SC 29615, USA

ARTICLE INFO

*Article history:*
Received 18 August 2010
Received in revised form 13 October 2010
Available online 21 November 2010

*Keywords:*
Contact
Curved beam
Extensibility
Shearing deformation
Timoshenko beam

ABSTRACT

Contact of a circular ring with a flat, rigid ground is considered using curved beam theory and analytical methods. Applications include tires, springs, and stiffeners, among others. The governing differential equations are derived using the principle of virtual work and the formulation includes deformations due to bending, transverse shear and circumferential extension. The three associated stiffness quantities, *EI*, *GA* and *EA*, respectively, remain as independent parameters in the differential equations. This allows the special cases such as an inextensible Timoshenko beam (*EI* and *GA*) or an extensible Euler beam (*EI* and *EA*) to be obtained directly by the appropriate limits. The effect of these three stiffness parameters on the contact pressure solution is studied, which shows how those fundamental parameters can be selected for the purpose of the application. Although the formulation is for small displacement theory, both radial and circumferential distributed loads are considered, which allows the pressure in the deformed state to be vertical rather than radial, which is shown to be important. Closed form expressions for all force and displacement quantities are obtained in terms of the angular location of the edge of contact, which must be determined numerically. Extensibility complicates the analytical expressions within the contact region, and a series solution is proposed in this case. A two-term asymptotic expression for the stiffness of the ring is determined analytically. Finally, all solutions are validated using the commercial finite element software ABAQUS, with attention to non-linear behavior and the range of validity of these solutions.



## 1. Introduction

Curved beams or bars can be encountered in several engineering applications, such as bridge structures, aerospace structures, tires, springs, pipes, stiffeners for shells, among others. Due to the wide and extensive use of curved beams, a great deal of interest has been shown in theory development and solutions of a variety of associated mechanics problems. While the first contributions can be traced back to the 19th century, many analytical and computational investigations involving both dynamic and static behavior have been done during the past three or four decades.

Most of the recent effort has focused on vibrations of circular arches that take axial extension and transverse shear deformation into account. For example, Tüfekçi and Arpaci (1998) determined the exact solution of free in-plane vibrations of circular arches of uniform cross-section including the effects of axial extension, transverse shear deformation and rotary inertia effects. Lin and Lee (2001) presented closed-form solutions for dynamic analysis of extensional circular Timoshenko beams with general elastic boundary conditions by using generalized Green function given by Lin (1998). Qatu (1993) developed a set of equations of motion for thin and thick laminated curved beams and obtained exact closed form natural frequencies for simply-supported curved beams. He also studied the effects of rotary inertia, shear deformation, curvature and thickness ratios, and material orthotropy on the natural frequencies. For additional references see Markus and Nanasi (1981), Laura and Maurizi (1987) and Chidamparam and Leissa (1993).

There has also been recent interest in static analysis of curved beams since the early contributions of Timoshenko (1955), in Chapter VI, who extended the linear bending theory of Euler–Bernoulli to curved beams and Flügge (1960), who established the stress analysis of shells and considered cylindrical and spherical shells in detail. Lim et al. (1997) presented the exact relationships between deflection and stress resultants of Timoshenko and Euler–Bernoulli curved beam when a transverse load acts at the beam centerline. Such results enable straightforward conversion of the "familiar" Euler–Bernoulli solutions into those of Timoshenko. Lin (1998) presented a Green's function approach to solve $n$th-order ordinary differential equations and applied the method to obtain the exact solution for static analysis of an extensible circular curved

* Corresponding author. Tel.: +1 864 656 0545.
*E-mail address:* JPAUL@CLEMSON.EDU (P.F. Joseph).

0020-7683/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2010.11.018