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Compressibility and shear compliance of spheroidal pores: Exact derivation via the Eshelby tensor, and asymptotic expressions in limiting cases

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ABSTRACT

We explicitly calculate the elastic compliance of a spheroidal pore in an isotropic solid, starting from Eshelby's tensor. The exact expressions found for the pore compressibility, P, and the shear compliance, Q, are valid for any value of the aspect ratio α, from zero (cracks) to infinity (needles). This derivation clarifies previous work on this problem, in which different methods were used in different ranges of α, or typographical errors were present. The exact expressions obtained for P and Q are quite complex and unwieldy. Simple expressions for both P and Q have previously been available for the limiting cases of infinitely thin-cracks ($\alpha = 0$), infinitely long-needles ($\alpha = \infty$), and spherical pores ($\alpha = 1$). We have now calculated additional terms in the asymptotic expansions, yielding relatively simple approximations for P and Q that are valid for crack-like pores having aspect ratios as high as 0.3, needle-like pores having aspect ratios as low as 3, and nearly spherical pores. Their relatively simple forms will be useful for incorporation into various schemes to estimate the effective elastic moduli.

1. Introduction

A spheroidal inclusion in an infinite, isotropic elastic solid provides a useful model to calculate elastic properties of rocks, ceramics, bones, or other porous materials. Indeed, the ellipsoid is the only three-dimensional shape amenable to analytical treatment, and exact expressions for the elastic field of an ellipsoidal inclusion were obtained by Eshelby (1957). The exact solutions for ellipsoids are, however, very unwieldy and expressed in terms of elliptic integrals, therefore they require to be computed numerically. Nevertheless, some recent papers have indeed used the ellipsoidal pore as the basis of their calculations (Markov et al., 2005; Giraud et al., 2008). However, most modelling efforts have utilised the *spheroid*, which is simply a degenerate ellipsoid with two axis of equal length, as it offers simpler expressions. In its various forms, a spheroid can represent a wide variety of pore shapes, such as thin cracks, cylinders, or spheres (Fig. 1).

Exact expressions for the elastic compliance of spheroidal pores, i.e., the pore compressibility P and the shear compliance Q, have been presented by Wu (1966) in terms of two invariants (T_{iijj}, T_{ijij}) of Wu's tensor $\mathbb{T}$, where $\mathbb{T}$ is a fourth-order "strain-concentration" tensor relating the strain in the inclusion to the homogeneous strain applied at infinity. Wu's derivation (based on Eshelby's results) is exact, but the expressions given in his paper contain several typographical errors. The correct results were presented later by Kuster and Toksoz (1974), but their derivation used a wave-scattering approach, and it is not entirely obvious that their results should be the same as those derived from Wu's tensor. Later, such expressions for (T_{iijj}, T_{ijij}) were also presented in Cheng and Toksoz (1979) or Berryman (1980), but they are based on the previous results of Kuster and Toksoz. Close-form expressions for P and Q, based on Eshelby' results, are given by Dunn and Ledbetter (1995), in the limit of a dry spheroid. Unfortunately, their expressions again contain some typographical errors. Kachanov et al. (2003) present expressions for the pore compliances using the $\mathbb{H}$-tensor formalism, which is completely equivalent to the $\mathbb{T}$-tensor one. For a spheroid in an isotropic medium, $\mathbb{H}$ is a *transversely isotropic* tensor, whose components are explicitly given as functions of Eshelby's components, and from which the pore compressibility and shear compliance can be extracted by some algebraic manipulations. However, their expressions contain also some typographical errors, which seem to start with an extraneous factor of 2 that appears outside of the parenthesised term in the first equation on p. 263. Note that the same errors were included previously in Shafiro and Kachanov (1997). Hence, it seems difficult to find in the literature exact expressions for the pore compliances of spheroids that are derived from Eshelby's results in a clear manner. Our first preliminary goal is therefore to explicitly derive the correct expressions for the pore compliances, starting from Wu's tensor (Section 2).

The expressions we find for the elastic compliances are exact and applicable for the entire range of spheroid aspect ratios, from zero (flat cracks) to infinity (long needles). But these expressions are too cumbersome to be easily interpreted, and are too complex

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