



## Controllability, uniqueness and existence of the incremental response: A mathematical criterion for elastoplastic constitutive laws

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### ABSTRACT

In this paper the onset of instabilities in elastoplastic materials is theoretically studied and a conceptual basis for understanding the physical implications of a loss of uniqueness and/or existence of the incremental response is provided. For this purpose, the concept of test controllability is reinterpreted and mixed stress–strain loading programmes are accounted for. A set of scalar indices, the moduli of instability, related with the inception of an unstable response is introduced and their dependency on the loading programme is explicitly illustrated. The paper shows that the use of these newly defined scalar measures provides support for an alternative definition for mechanical stability, which is closely related with the mathematical notions of existence and uniqueness of the predicted incremental response. In the final section, some mathematical properties of the moduli of instability are discussed, suggesting a novel reinterpretation of other well established theories and providing additional tools for the future application of the proposed framework.

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### 1. Introduction

The proper definition of the concept of failure still represents a fundamental and unsolved mechanical issue. According to a common engineering perspective, the identification of failure conditions for a solid material requires the assessment of a limit locus at which failure occurs. This apparently unambiguous concept, however, becomes less neat when complex natural and engineered materials are considered. A notable example is represented by soils, rocks and concrete (often referred to as *geomaterials*). In these cases, in fact, the material can suffer a broad range of unstable responses even within an apparently safe domain. Sophisticated modeling tools and efficient computational techniques are therefore required in order to capture the experimental evidence and to implement these models in numerical codes for solving boundary value problems.

During the last few decades, several theoretical approaches have been proposed to deal with material instability at a constitutive level (Darve and Chau, 1987; Nova, 1989, 1994; Darve, 1994; Chambon, 2005; Borja, 2006). Most of these theories tried to generalize the concept of failure, giving it a much broader meaning (Nova, 2003). Different strategies were used to identify unstable conditions: Imposimato and Nova (1998) discussed for instance the vanishing of suitable mathematical operators governing the material response, Nicot et al. (2007) studied the sign of the

second-order work, while Sibille et al. (2007) combined several modeling techniques. In particular, Imposimato and Nova (1998) introduced the notion of test controllability, showing that, depending on the particular loading programme imposed to the soil specimen, critical conditions can arise even within the hardening regime. This concept provided an enlightening interpretation of the Hill's stability criterion (1958), suggesting an alternative mechanical meaning for it.

Although these approaches are based on consistent premises, their use becomes less powerful when more general initial and loading conditions are accounted for. The concept of second-order work, in fact, does not provide any insight into the role of control parameters (e.g., in case of mixed stress–strain loading paths). Moreover, since the theory of test controllability has been developed and widely applied to treat instabilities in the hardening regime, it can present disadvantages when this particular case is abandoned (e.g., in rock-like materials, that are often characterized by a marked strain-softening response).

This paper stems from the necessity of providing a consistent theoretical reference capable of assessing the stability of a given stress state in a more general and flexible manner. Therefore, an alternative approach for addressing geomaterial instability is here presented, with the main goal of explaining material instability under a different perspective. For the sake of simplicity, the following discussion is restricted to the case of rate-independent elastoplastic constitutive laws (also referred to as models with two “tensorial zones” (Nicot and Darve, 2005)). Even though this choice does not include important issues (e.g., the effects of full incremental

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