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# A C2 continuous approximation to the Mohr–Coulomb yield surface

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In spite of the development of more sophisticated constitutive models for soil, the Mohr–Coulomb yield criterion remains a popular choice for geotechnical analysis due to its simplicity and ease of use by practising engineers. The implementation of the criterion in finite element programs, however, presents some numerical difficulties due to the gradient discontinuities which occur at both the edges and the tip of the hexagonal yield surface pyramid. Furthermore, some implicit techniques utilising consistent tangent stiffness formulations are unable to achieve full quadratic convergence as the yield criteria is not C2 continuous. This paper extends the previous work of Abbo and Sloan (1995) through the introduction of C2 continuous rounding of the Mohr–Coulomb yield surface in the octahedral plane. This approximation, when combined with the hyperbolic approximation in the meridional plane (Abbo and Sloan, 1995), describes a yield surface that is C2 continuous at all stress states. The new smooth yield surface can be made to approximate the Mohr–Coulomb yield function as closely as required by adjusting only two parameters, and is suitable for consistent tangent stiffness formulations.



## 1. Introduction

The Mohr–Coulomb yield criterion provides a relatively simple model for simulating the plastic behaviour of soil. Other more sophisticated constitutive models for predicting the behaviour of soil have been developed over the past three decades, however the complexity of these models, as well as the additional testing required to determine the various soil parameters involved, minimises their utility for practising geotechnical engineers. The Mohr–Coulomb yield function is also of importance to finite element researchers and practitioners as it forms the basis of many analytical solutions. These analytical solutions serve as crucial benchmarks for validating numerical algorithms and software.

In three-dimensional principal stress space, the Mohr–Coulomb yield criterion is a hexagonal pyramid whose central axis lies along the hydrostatic axis as shown in Fig. 1(a). The implementation of the Mohr–Coulomb yield surface in finite element programs is complicated by the presence of the vertices at the tip and along the sides of the Mohr–Coulomb pyramid. At these vertices, the yield function is not C1 continuous (i.e., first derivative is not continuous) let alone C2 continuous (i.e, second-derivative is not continuous). It is necessary to address these singularities because stress states lying at, or near, the vertices are often encountered in practice. One approach to overcoming the computational difficulties posed by the vertices is to consider the Mohr–Coulomb surface as six separate planar yield surfaces and implement the constitutive law as a multi-surface yield function using the formulation of Koiter (1953) (e.g. Ristinmaa and Tryding, 1993; Clausen et al., 2006). The approach used in this paper is to derive a smooth approximation to the yield surface that eliminates the sharp vertices by rounding the corners of the Mohr–Coulomb yield surface. The rounding is derived so that it closely approximates the true yield surface yet provides C2 continuity.

Mathematically the Mohr–Coulomb yield criterion can be described in terms of the principal stresses ($\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$) as

$$F = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi = 0 \tag{1}$$

in which $c$ and $\phi$ represent the cohesion and friction angle of the soil and tensile stresses are considered positive. A more convenient form of the criterion, which avoids explicit calculation of principal stresses, was proposed by Nayak and Zienkiewicz (1972). They expressed the criterion as a function of the three stress invariants ($\sigma_m, \bar{\sigma}, \theta$) (see Appendix A) as

$$F = \sigma_m \sin\phi + \bar{\sigma}K(\theta) - c\cos\phi = 0 \tag{2}$$

in which

$$K(\theta) = \cos\theta - \frac{1}{\sqrt{3}}\sin\phi\sin\theta \tag{3}$$

is a function controlling the shape of the surface in the octahedral plane (the plane orthogonal to the hydrostatic axis).

The gradient discontinuities at the tip and along the sides of the hexagonal pyramid can be considered separately by studying the meridional and octahedral sections of the yield surface. The

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