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Hill's class of compressible elastic materials and finite bending problems: Exact solutions in unified form

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ABSTRACT

Hill (1978) proposed a natural extension of Hooke's law to finite deformations. With all Seth-Hill finite strains, Hill's natural extension presents a broad class of compressible hyperelastic materials over the whole deformation range. We show that a number of known Hookean type finite hyperelasticity models are included as particular cases in Hill's class and that Bell's and Ericksen's constraints may be derived as natural consequences from Hill's class subjected to internal constraints. Also we present a unified study of finite bending problems for elastic Hill materials. To date exact results are available for certain particular classes of compressible elastic materials, which do not cover Hill's class. Here, with a novel idea of circumventing the strong nonlinearity we show that it is possible to derive exact solutions in unified form for the whole class of elastic Hill materials. Reduced results are also given for cases subjected to internal constraints.

1. Introduction

In a contribution to solid mechanics at finite deformations, Hill (1978) proposed an extension of Hooke's law from infinitesimal to finite deformations, which "might be regarded as a natural generalization of Hooke's law" (Hill, 1978, p. 49). With work-conjugate stress–strain pairs, Hill's extension actually defines a broad class of compressible hyperelastic materials of Hookean type, in conjunction with Seth–Hill or Doyle–Ericksen class of finite strain measures. Each of elastic Hill materials retains the structure of Hooke's law, with the same elastic constants specified at infinitesimal strain as in Hooke's law. Then, nonlinearity effects at finite deformations are incorporated into and characterized by the definition of finite strain measure. Several known finite elasticity models of Hookean type (see, e.g., Truesdell, 1952; Truesdell and Toupin, 1960; Truesdell and Noll, 1965) are naturally included as a few particular cases of finite strain measure, such as Green strain, Almansi strain, Cauchy–Biot strain, Swainger strain, Hencky's logarithmic strain, etc. Details will be given later on.

Finite bending deformation of a rectangular block is regarded as a representative, significant non-homogeneous finite deformation mode. In this article, we intend to make a unified study of finite bending problems of rectangular blocks made of elastic Hill materials with any given lateral stretch normal to the bending plane. With a novel method, we shall derive closed-form exact solutions in unified form for the whole class of Hill's compressible elastic materials.

It appears that modern studies in obtaining exact solutions for finite deformation problems of isotropic elastic materials originated from Rivlin's systematic treatment for rubber elasticity (cf., e.g., Rivlin, 1948a,b,c,d, 1949a,b). As shown in Rivlin's classical results, the incompressibility condition or constraint may lead to substantial reduction to the strong nonlinear coupling. Since Rivlin's pioneering works, numerous results have been derived for closed-form solutions of finite deformation problems for incompressible hyperelastic materials; refer to, e.g., Doyle and Ericksen (1956), Green and Zerna (1960), and Wang and Truesdell (1973) for classical results and Ogden (1984), Beatty (1987, 2001), and Fu and Ogden (2001), etc. for recent results. In recent years, many efforts have been made to obtain exact solutions for finite elastic deformation problems of compressible materials. Without uncoupling from the incompressibility condition, results were usually derived for particular forms of elastic energy functions. With a general form of series expansion in terms of principal stretches, Ogden (1972a,b, 1976, 1978) made a systematic study of both incompressible and compressible deformations. Now numerous results for exact closed-form solutions were obtained for various finite homogeneous and inhomogeneous deformation problems with

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