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Bipolar coordinates, image method and the method of fundamental solutions for Green's functions of Laplace problems containing circular boundaries

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ABSTRACT

Green's functions of Laplace problems containing circular boundaries are solved by using analytical and semi-analytical approaches. For the analytical solution, we derive the Green's function using the bipolar coordinates. Based on the semi-analytical approach of image method, it is interesting to find that the two frozen images for the eccentric annulus using the image method are located on the two foci in the bipolar coordinates. This finding also occurs for the cases of a half plane with a circular hole and an infinite plane containing two circular holes. The image method can be seen as a special case of the method of fundamental solutions, which only at most four unknown strengths are required to be determined. The optimal locations of sources in the method of fundamental solutions can be captured using the image method and they are dependent on the source location and the geometry of problems. Three illustrative examples were demonstrated to verify this point. Results are satisfactory.

1. Introduction

A number of physical and engineering problems governed by the Lapalce equation in two independent variables, e.g., steady-state heat conduction, electrostatic potential and fluid flow, were solved using the conformal mapping to obtain an analytical solution. Besides, we can formulate the same problems using special curvilinear coordinates to obtain a solution, e.g., bipolar coordinates and elliptic coordinates. Carrier and Pearson [1] employed the bilinear transformation of conformal mapping to solve certain kinds of potential problems. An eccentric case was mapped to an annular domain through a bilinear transformation. For a polygonal shape, it can also be mapped to a regular region using the Schwarz–Christoffel transformation [2]. For a regular geometry, it is easy to solve the Laplace problem using the polar or Cartesian coordinates. Muskhelishvili [3] gave us a detailed description how an eccentric annulus can be mapped into a concentric annulus using a simple form of linear fractional transformation. Chen and Weng [4] also used a similar method to solve eccentric annulus problems. Although a bilinear transformation was used, the mapping functions were not exactly the same between the one of Carrier and Pearson [1] and that of Muskhelishvili [3]. Problems of eccentric annulus, a half plane with a circular hole or an infinite plane containing two circular holes were usually solved by using the bipolar coordinates to derive the analytical solution [5]. Ling [6], Timoshenko and Goodier [7], and Lebedev et al. [8] all presented an analytic solution by using the bipolar coordinates for the torsion problem of an eccentric bar. However, the mapping functions were not exactly the same. One is a cotangent function [6], another is a hyperbolic tangent function [8] and the other is a hyperbolic cotangent function [7]. After the bipolar coordinate system is introduced, the problem of special domain can be solved by using the separation of variables. Although Carrier and Pearson [1], Muskhelishvili [3], Ling [6], Timoshenko and Goordier [7] have solved the eccentric Laplace problems, their approaches are very similar, but not identical. Chen et al. [9] found that all the above-mentioned approaches can be unified after suitable transformations, translation, rotation and taking *log* in the conformal mapping. However, we will focus on Green's function instead of BVP without sources [10] in this paper.

Green's function has been studied and applied in science and engineering by mathematicians as well as engineers, respectively [11]. A computer-friendly solution for the potential generated by a point source in the ring-shaped region was studied by Melnikov and Arman [12]. In order to derive Green's function, Thomson [13] proposed the concept of reciprocal radii to find the image source to satisfy the homogeneous Dirichlet boundary condition using the image method. Greenberg [14] and Riley et al. [15] employed a trick to satisfy the boundary condition for two special points, then the image location can be determined. Chen and Wu [16] proposed a natural and logical way to find the location of image and the strength by employing the degenerate kernel. The image method is a classical approach for constructing Green's function. In certain cases, it is possible to obtain the exact solution for a concentrated source in a bounded domain through

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