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The nature of Hopf bifurcation for the Gompertz model with delays

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ABSTRACT

In this paper, we study the influence of time delays on the dynamics of the classical Gompertz model. We consider the models with one discrete delay introduced in two different ways and the model with two delays which generalise those with one delay. We study asymptotic behaviour and bifurcations with respect to the ratio of delays $\bar{\tau} = \tau_1/\tau_2$. Our results show that in such model with two delays there is only one stability switch and for a threshold value of bifurcation parameter, Hopf bifurcation (HB) occurs. However, the type of HB, and therefore its stability (i.e. stability of periodic orbits arising due to it), strongly depends on the magnitude of $\bar{\tau}$. The function describing stability of HB is periodic with respect to $\bar{\tau}$. Within one period of length 4 five changes of HB stability are observed.

We also introduce the second model with two delays which has a better biological interpretation than the first one. In that model several stability switches can occur, depending on the model parameters.

We illustrate analytical results on the example of tumour growth model with parameters estimated on the basis of experimental data.

1. Introduction

The Gompertz equation is one of the most popular non-linear models for self-limiting cell population growth. For the first time it was proposed by Benjamin Gompertz in 1825 [1]. He derived the algebraic formula and used it in the context of actuarial statistics. Later in 1932, Winsor modified it and found that it provides a good empirical description of decelerating tumour growth; see [2]. In 1964, the same model was used by Laird in the description of tumour growth [3] and later in 1965, [4], for the first time Laird fitted the experimental data for a variety of primary and transplanted tumours of the mouse, rat and rabbit to the Gompertz curve. In the same year, Laird et al. [5] showed that the Gompertz model, due to its ability to exhibit exponential retardation, could describe the normal growth of an organism such as the guinea pig over an incredible 10 000-fold range of the growth. Since that time the Gompertz equation is often used in the formulation of equations describing the population dynamics and to describe the inner growth of tumour. It has been widely incorporate to models describing the process of angiogenesis (see for example [6–8]) and to investigate the optimal treatment protocols (see [9–12]). However, it has been recognised that in the case of tumour growth such models as the Gompertz equation should be combined with an exponential growth for small tumours; see [13], the discussion on the Gompertz and Gomp-ex model and the references therein, and more recent publications on the Gompertz model and its modifications, [14,15] and the references therein.

On the other hand, time delays are often introduced to the models to better reflect reality of considered processes; see [16,17] in the context of population dynamics, [18–26] in the description of the immune system, [27–36] in the context of avascular tumour growth, [37,38,7,8,39] in the description of tumour angiogenesis, [40–42] for tumour-immune

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