C^*-algebras of Toeplitz and composition operators

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Abstract

We investigate the unital C^*-algebras generated by an irreducible Toeplitz operator T_ψ and one or more composition operators C_φ induced by linear-fractional self-maps φ of the unit disk acting on the Hardy space H^2, modulo the ideal of compact operators $K(H^2)$. For automorphism symbol φ, we compare this algebra with the one generated by the shift operator T_z and a composition operators.

Keywords: the unilateral shift operator, Toeplitz operator, composition operator, linear-fractional map, automorphism of the unit disk.
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1 Introduction

The Hardy space $H^2 = H^2(\mathbb{D})$ is the collection of all analytic functions f on the open unit disk $\mathbb{D}$ satisfying the norm condition $\|f\|^2 := \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty$. For any analytic self-map φ of the open unit disk $\mathbb{D}$, a bounded composition operator on H^2 is defined by

$$C_\varphi : H^2 \to H^2, \quad C_\varphi(f) = f \circ \varphi.$$

If $f \in H^2$, then the radial limit $f(e^{i\theta}) := \lim_{r\to 1} f(re^{i\theta})$ exists almost everywhere on the unit circle $\mathbb{T}$. Hence we can consider H^2 as a subspace of $L^2(\mathbb{T})$. Let ϕ is a bounded measurable function on $\mathbb{T}$ and P_{H^2} be the orthogonal projection of $L^2(\mathbb{T})$ (associated with normalized arc-length measure on $\mathbb{T}$) onto H^2. The Toeplitz operator T_ϕ is defined on H^2 by $T_\phi f = P_{H_2}(\phi f)$ for all $f \in H^2$. Coburn in [2] shows that the quotient of the unital C^*-algebra $C^*(T_z)$ generated by the unilateral shift operator T_z on the ideal of compact operators $\mathfrak{K} = K(H^2)$ is $*$-isomorphic to $C(\mathbb{T})$, and determines essential spectrum of Toeplitz operators with continuous symbol. Recently the unital C^*-algebra generated by the shift operator T_z and the composition operator C_φ for a linear-fractional self-map φ of $\mathbb{D}$ is studied. For a linear-fractional self-map φ on $\mathbb{D}$, if $\|\varphi\|_\infty < 1$ then C_φ is a compact operator on H^2. Therefore we consider those linear-fractional self-maps φ which satisfy $\|\varphi\|_\infty = 1$. If moreover φ is an automorphism of $\mathbb{D}$, then $C^*(T_z, C_\varphi)/\mathfrak{K}$ is $*$-isomorphic to the crossed products $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ or $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}_n$ [4]. When φ is not an automorphism there are three different cases:

(i) φ has only one fixed point γ which is on the unit circle $\mathbb{T}$ (i.e. φ is a parabolic map) [7]. In this case, $C^*(T_z, C_\varphi)/\mathfrak{K}$ is a commutative C^*-algebra isomorphic to $C_\gamma(\mathbb{T}) \oplus C_0([0, 1])$, where $C_\gamma(\mathbb{T})$ is the set of functions in $C(\mathbb{T})$ vanishing at $\gamma \in \mathbb{T}$.

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