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Closed non-vanishing ideals in $C_B(X)$

M. R. Koushesh
Isfahan University of Technology

Abstract

Let X be a completely regular space. For a closed non-vanishing ideal H in $C_B(X)$ we construct the spectrum $\mathfrak{sp}(H)$ of H as a subspace of the Stone–Čech compactification of X. The known construction of $\mathfrak{sp}(H)$ will then enable us to derive certain properties of $\mathfrak{sp}(H)$ which are not generally expected to be easily deducible from the standard Gelfand theory.

This paper is a rather self-contained extract from the research monograph [M. R. Koushesh, *Ideals in* $C_B(X)$ *arising from ideals in* X, 53 pp.] available as the arXiv preprint `arXiv:1508.07734 [math.FA]`, to which the reader may also be referred to.

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1 Introduction

Throughout this paper by a *space* we will mean a *topological space*.

Let X be a completely regular space. Let $C_B(X)$ be the algebra of all complex valued continuous bounded mappings on X equipped with the supremum norm. Also, let $C_0(X)$ be the subset of $C_B(X)$ consisting of all f which vanish at infinity (i.e., $|f|^{-1}([\epsilon,\infty))$ is compact for each $\epsilon > 0$). A subset H of $C_B(X)$ is said to be *non-vanishing* if for each x in X there is some h in H such that $h(x) \neq 0$.

The commutative Gelfand–Naimark theorem states that every commutative C^*-algebra A is isometrically $*$-isomorphic to $C_0(Y)$ for some locally compact Hausdorff space Y. Such a space Y is necessarily unique (up to homemorphism) by the Banach–Stone theorem and is identical to the spectrum of A. Here, using purely topological arguments, we prove that a closed non-vanishing ideal H of $C_B(X)$ is isometrically isomorphic to $C_0(Y)$ for a locally compact space Y. This in particular re-proves the commutative Gelfand–Naimark theorem in its special case. We construct Y as a subspace of the Stone–Čech compactification of X. The known construction of Y will then enable us to study it deeper and derived results which are not generally expected to be easily deducible from the standard Gelfand theory.

This paper is an extract from the research monograph [10]. However, it is rather self-contained, as it contains a complete proof for its main result (Theorem 2.7). For proofs of the remaining results (Theorems 2.9 and 2.10) we refer the interested reader to the original preprint [10]. (See [6]–[8] for further related results.)