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Set-theoretic methods of homological algebra and their applications to module theory

Jan Trlifaj*
Charles University, Prague

Abstract

We present some of the recent tools of set-theoretic homological algebra together with their applications, notably to the approximation theory of modules, and to (infinite dimensional) tilting.

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1 Introduction

A major topic of module theory concerns existence and uniqueness of direct sum decompositions. Positive results provided by the Krull-Remark-Schmidt-Azumaya theorems, the Faith-Walker Theorem, and Kaplansky theorems, form the cornerstones of the classic theory. However, there are a number of important classes of (not necessarily finitely generated) modules to which the theory does not apply, because their modules do not decompose into (possibly infinite) direct sums of indecomposable, or small, submodules.

While such direct sum decompositions are rare, there do exist more general structural decompositions that are almost ubiquitous. The point is to replace direct sums by transfinite extensions. For example, taking direct sums of copies of the group $\mathbb{Z}_p$, one obtains all $\mathbb{Z}_p$-modules whose sole isomorphism invariant is the vector space dimension. In contrast, transfinite extensions of copies of $\mathbb{Z}_p$ yield the much richer class of all abelian p-groups whose isomorphism invariants are known basically only in the totally-projective case (the Ulm-Kaplansky invariants).

Starting with the solution of the Flat Cover Conjecture [5], numerous classes $\mathcal{C}$ of modules have been shown to be deconstructible, that is, expressible as transfinite extensions of small modules from $\mathcal{C}$. Basic tools for deconstruction come from set-theoretic homological algebra and originate in abelian group theory [6], but have since been expanded and generalized to module categories, and even beyond that setting.

Each deconstructible class is precovering, so it provides for approximations of modules. By choosing appropriately the class $\mathcal{C}$, one can tailor these approximations to the needs of various particular structural problems, cf. [12].

*Speaker