On Fuglede–Putnam Theorem

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Abstract

For operators A and B, let $\mathrm{Com}(A, B)$ stand for the set of operators X such that $AX = XB$. A pair (A, B) is said to have the (FP)-property if $\mathrm{Com}(A, B) \subseteq \mathrm{Com}(A^*, B^*)$. Let $\tilde{C}$ denote the Aluthge transform of a bounded linear operator C, (i) if A and B are invertible operators and (A, B) has the (FP)-property, then so is $(\tilde{A}, \tilde{B})$; (ii) if A and B are invertible and $U \pm \mathrm{i}I$ and $V \pm \mathrm{i}I$ are all invertible and $(\tilde{A}, \tilde{B})$ has the (FP)-property, then so is (A, B); (iii) if $U^2|A|U$ has the (FP)-property and $A^3 = I$, then A is a unitary operator.

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1 Introduction

Let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on (separable) complex Hilbert spaces $\mathcal{H}$, and let $I \in \mathbb{B}(\mathcal{H})$ be the identity operator. A subspace $\mathcal{K} \subseteq \mathcal{H}$ is said to reduce $A \in \mathbb{B}(\mathcal{H})$ if $A\mathcal{K} \subseteq \mathcal{K}$ and $A^*\mathcal{K} \subseteq \mathcal{K}$. Let $\mathbb{K}(\mathcal{H})$ denote the two-sided ideal of all compact operators on $\mathcal{H}$. For $p > 0$, an operator A is called p-hyponormal if $(A^*A)^p \geq (AA^*)^p$. If A is an invertible operator satisfying $\log(A^*A) \geq \log(AA^*)$, then it is called log-hyponormal. If $p = 1$, then A is said to be hyponormal. If A is invertible and p-hyponormal then A is called log-hyponormal.

Let $A = U|A|$ be the polar decomposition of A. It is known that if A is invertible then U is unitary and $|A|$ is also invertible. The Aluthge transform $\tilde{A}$ of A is defined by $\tilde{A} := |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. This notion was first introduced by Aluthge [1] and is a powerful tool in the operator theory. An interesting application of Aluthge transform deals with an generalizing the Fuglede–Putnam theorem [3]. Let $A, B \in \mathbb{B}(\mathcal{H})$ and . For such pair (A, B), denote by $\mathrm{Com}(A, B)$ the set of operators $X \in \mathbb{B}(\mathcal{H})$ such that $AX = XB$. A pair (A, B) is said to have the (FP)-property if $\mathrm{Com}(A, B) \subseteq \mathrm{Com}(A^*, B^*)$. The Fuglede–Putnam theorem is well-known in the operator theory. It asserts that for any normal operators A and B, the pair (A, B) has the (FP)-property. First Fuglede proved it in the case when $A = B$ and then Putnam proved it in a general case; see [4]. There exist many generalizations of this theorem which most of them go into relaxing the normality of A and B; see [4] and references therein. The two next Theorems are concerned the Fuglede–Putnam theorem.

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