Multilinear mappings on matrix algebras

Mahdi Dehghani*
University of Kashan

Mohsen Kian
University of Bojnord

Abstract

We investigate the notion of positive multilinear mappings on matrix algebras. Some matrix inequalities including positive multilinear mappings are introduced.

Keywords: positive multilinear mapping, Jensen inequality, positive matrix, matrix convex function

Mathematics Subject Classification [2010]: Primary 15A69 ; Secondary 47A63,47A64, 47A56.

1 Introduction

Let $\mathcal{M}_n := \mathcal{M}_n(\mathbb{C})$ be the C^*-algebra of all $n \times n$ complex matrices with identity matrix I. A linear map $\Phi : \mathcal{M}_q \to \mathcal{M}_p$ is called positive if $\Phi(A) \geq 0$ in $\mathcal{M}_p$, whenever $A \geq 0$ in $\mathcal{M}_q$. Positive linear mapppings on C^*-algebras and their related operator inequalities are well-known and have been studied by many mathematicians; see e.g., [1, 2, 4] and the references therein. Positive linear mappings have been used to characterize matrix convex functions. A continuous real function $f : J \to \mathbb{R}$ is said to be matrix convex if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$ for all $\lambda \in [0,1]$ and all hermitian matrices A, B with eigenvalues in J. It is well-known that a continuous real function $f : J \to \mathbb{R}$ is matrix convex if and only if

$$f(\Phi(A)) \leq \Phi(f(A)) \tag{1}$$

for every unital positive linear mapping Φ and every hermitian matrix A with spectrum in J. The inequality (1) is known as the Choi-Davis-Jensen inequality, see [2, 4].

The notion of positive linear mappings is introduced also for maps of several variables. Let $\mathcal{A}_k$, $k = 1, \cdots, n$ and $\mathcal{B}$, be C^*-algebras. A map $\Phi : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{B}$ is called to be positive multilinear if, it is linear in each of its variable and for every positive elements $a_k \in \mathcal{A}_k$, $k = 1, \cdots, n$, $\Phi(a_1, \cdots, a_n)$ is positive in $\mathcal{B}$ [5].

It is known that if A and B are positive matrices, then so is their Hadamard (Schur) product, $A \circ B$. The same is true for tensor product, $A \otimes B$. Moreover, the mapping $(A, B) \to A \otimes B$ is also linear in each of its variables. So if we define $\Phi : \mathcal{M}_q^2 \to \mathcal{M}_p$ by $\Phi(A, B) = A \otimes B$, then Φ is multilinear and positive in the sense that $\Phi(A, B)$ is positive, whenever A, B are positive.

*Speaker