On Induced Closed Subobjects by certain morphism classes*

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Abstract

In this article, we start with a collection $\mathcal{M}$ of morphisms of a small category $\mathcal{X}$, that satisfies certain conditions and we construct an universal closure operation by general method. Next describe closed subobjects relative to this universal closure by a different view point.

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1 Introduction

Throughout this article we let $\mathcal{X}$ be a small category and $\mathcal{M}$ be a set of morphisms of $\mathcal{X}$. The collection, $\mathcal{X}_1/x$, of all the $\mathcal{X}$-morphisms with codomain x is a preordered class by the relation $f \leq g$ if there exists a morphism h such that $f = g \circ h$. The equivalence relation generated by this preorder is $f \sim g$ if $f \leq g$ and $g \leq f$. For a class $\mathcal{M}$ of $\mathcal{X}$-morphisms, we write $f \sim \mathcal{M}$ whenever $f \sim m$ for some $m \in \mathcal{M}$. We say $\mathcal{M}$ is saturated provided that $f \in \mathcal{M}$ whenever $f \sim \mathcal{M}$.

Denoting the domain and codomain of a morphism f by $d_0 f$ and $d_1 f$ respectively, recall that a sieve in $\mathcal{X}$, [6], generated by one morphism f is called a principal sieve and is denoted by $\langle f \rangle$ and for a sieve S on x and a morphism f with $d_1 f = x$, $S \cdot f = \{g : f \circ g \in S\}$. We say sieve S is an k-sieve provided that minimum number of morphisms generates S is equal to k.

For a class $S \subseteq \mathcal{X}_1/x$, and a morphism f with codomain x, the class of all the maximal elements w in $\mathcal{X}_1/d_0 f$ satisfying $f \circ w \leq s$ for some $s \in S$ is denoted by $(f \Rightarrow S)$. Obviously for a sieve S on x, see [6], $(f \Rightarrow S)$ is just the class of maximal elements of $S \cdot f$.

We say $\mathcal{M}$ has $\mathcal{X}$-pullbacks if the pullback, $f^{-1}(m)$, of each $m \in \mathcal{M}$ along each $f \in \mathcal{X}$ exists and belongs to $\mathcal{M}$.

Definition 1.1. A class $\mathcal{M}$ of $\mathcal{X}-$morphisms is said to satisfy the principality property, if for each x, $f \in \mathcal{X}_1/x$ and $m \in \mathcal{M}/x$, $(f \Rightarrow \langle m \rangle) \subseteq \mathcal{M}/d_0 f, card((f \Rightarrow \langle m \rangle)) = 1$.

If $\mathcal{X}_1$ satisfies the principality property, the mapping $P^n : \mathcal{X}^{op} \to Set$ with $P^n(x) = \{\langle f_1, f_2, ..., f_n \rangle | \text{for all } 1 \leq i \leq n, f_i \in \mathcal{X}_1/x\}$ and for $f : y \to x$, $P^n(f) : P^n(x) \to P^n(y)$ the function taking $\langle g_1, g_2, ..., g_n \rangle$ to $\langle g_1, g_2, ..., g_n \rangle \cdot f$, is a functor. Also the mapping $P : \mathcal{X}^{op} \to Set$ with $P(x) = \{S \in \Omega(x) \mid S \text{ is finitely generated sieve on } x\}$ and for $f : y \to x$, $P(f) : P(x) \to P(y)$ the function taking S to $S \cdot f$, is a functor.

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