A survey of simplicial cohomology for semigroup algebras

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Abstract

In this survey, we investigate the higher simplicial cohomology groups of the convolution algebra $\ell^1(S)$ for various semigroups S. The classes of semigroups considered are semilattices, Clifford semigroups, regular Rees semigroups and the additive semigroups of integers greater than a for some integer a. Our results are of two types: in some cases, we show that some cohomology groups are 0, while in some other cases, we show that some cohomology groups are Banach spaces.

1 Introduction

In this talk, we investigate the higher simplicial cohomology groups of the convolution algebra $\ell^1(S)$ for various semigroups S. Our results are of two types: in some cases, we show that some cohomology groups are 0, while in some other cases, we show that some cohomology groups are Banach spaces.

First we explain the general idea for showing that a cohomology group is a Banach space. Let $\delta : C^n(\mathcal{A}, \mathcal{X}) \longrightarrow C^{n+1}(\mathcal{A}, \mathcal{X})$ be the boundary map. Then $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is a Banach space if and only if the range of δ is closed, which is the case if and only if δ is open onto its range, that is there exists a constant K such that if $\psi = \delta(\phi)$ is such that $\|\psi\| < 1$ then there exists $\phi_1 \in C^n(\mathcal{A}, \mathcal{X})$ such that $\|\phi_1\| < K$ and $\psi = \delta(\phi_1)$.

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{A}'$ be a Banach $\mathcal{A}$-bimodule in the usual way. An n-cochain is a bounded n-linear map T from $\mathcal{A}$ to $\mathcal{A}'$, which we denote by $T \in C^n(\mathcal{A}, \mathcal{A}')$. The map $\delta^n : C^n(\mathcal{A}, \mathcal{A}') \longrightarrow C^{n+1}(\mathcal{A}, \mathcal{A}')$ is defined by

$$\begin{aligned}(\delta^n T)(a_1, \dots, a_{n+1})(a_0) &= T(a_2, a_3, \dots, a_{n+1})(a_0 a_1)\\ &\quad - T(a_1 a_2, a_3, \dots, a_{n+1})(a_0)\\ &\quad + T(a_1, a_2 a_3, a_4, \dots, a_{n+1})(a_0) + \dots\\ &\quad + (-1)^n T(a_1, \dots, a_{n-1}, a_n a_{n+1})(a_0)\\ &\quad + (-1)^{n+1} T(a_1, \dots, a_n)(a_{n+1} a_0)\,.\end{aligned}$$

The n-cochain T is an *n-cocycle* if $\delta^n T = 0$ and it is an *n-coboundary* if $T = \delta^{n-1} S$ for some $S \in C^{n-1}(\mathcal{A}, \mathcal{A}')$. The linear space of all n-cocycles is denoted by $\mathcal{Z}^n(\mathcal{A}, \mathcal{A}')$, and the linear space of all n-coboundaries is denoted by $\mathcal{B}^n(\mathcal{A}, \mathcal{A}')$. We also recall that $\mathcal{B}^n(\mathcal{A}, \mathcal{A}')$ is included in $\mathcal{Z}^n(\mathcal{A}, \mathcal{A}')$ and that the n^{th} simplicial cohomology group $\mathcal{H}^n(\mathcal{A}, \mathcal{A}')$ is defined by the quotient

$$\mathcal{H}^n(\mathcal{A}, \mathcal{A}') = \frac{\mathcal{Z}^n(\mathcal{A}, \mathcal{A}')}{\mathcal{B}^n(\mathcal{A}, \mathcal{A}')}.$$