Real orthogonal eigenvalue decomposition of symmetric normal matrices

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Abstract

We propose an algorithm for eigenvalue decomposition of symmetric normal complex matrices via real orthogonal transformations. This algorithm answers positively to the open question which is raised in [M. Ferranti, R. Vandebril, *Computing eigenvalues of normal matrices via complex symmetric matrices*, J. Comput. Appl. Math., vol. 259, (2014), part A, 281-293].

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1 Introduction

There are various well-known methods for finding eigenpairs of complex matrices. Most of these methods are based on a two-step approach, first the original matrix is transformed to a unitary similar matrix of suitable shape, e.g. tridiagonal or Hessenberg matrix and then using standard methods like QR-methods, divide-and-conquer, etc. (see[3]) to compute the eigenvalue of a matrix. Though these two-step methods reduced the cost, but some of the properties of the original matrix can be neglected in these procedure. For example, when a symmetric normal matrix transformed to a tridiagonal matrix, the transformed matrix may not be normal anymore. In fact, a matrix is normal and symmetric if and only if it admits a real orthogonal eigenvalue decomposition [4], i.e. there are a real orthogonal matrix Q and a diagonal matrix Λ for a symmetric normal matrix A such that $A = Q\Lambda Q^T$. In this paper, we propose an algorithm for eigenvalue decomposition of any symmetric normal matrix A using only real orthogonal transformations.

Theorem 1.1. *[?] Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$ be given and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{y}\|_2 > 0$. If $\boldsymbol{x} = e^{i\theta}\boldsymbol{y}$ for some θ, let $U(\boldsymbol{x},\boldsymbol{y}) = e^{i\theta} I_n$; otherwise, let $\phi \in [0, 2\pi)$ be such that $\boldsymbol{y}^*\boldsymbol{x} = e^{i\phi}|\boldsymbol{y}^*\boldsymbol{x}|$ (take $\phi = 0$ if $\boldsymbol{y}^*\boldsymbol{x} = 0$), let $\omega = e^{i\phi}\boldsymbol{y} - \boldsymbol{x}$ and let $U(\boldsymbol{x},\boldsymbol{y}) = e^{i\phi} U_\omega$, in which $U_\omega = I - 2(\omega^*\omega)^{-1}\omega\omega^*$ is a Housholder matrix. . Then U is unitary and $U(\boldsymbol{x},\boldsymbol{y})\boldsymbol{y} = \boldsymbol{x}$.*

Theorem 1.2. *[?] Let $A \in M_n(\mathbb{C})$ be partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, in which A_{11} and A_{22} are square. Then A is normal if and only if A_{11} and A_{22} are normal and $A_{12} = 0$.*

Lemma 1.3. *[?] Let $\mathcal{N} \subset M_n(\mathbb{C}^n)$ be a commuting family of matrices, then some nonzero vector in $\mathbb{C}^n$ is an eigenvalue of every $A \in \mathcal{N}$.*