On split Clifford algebras with involution in characteristic two

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Abstract

In characteristic two, the involutions on split Clifford algebras induced by the involutions of orthogonal group are investigated. Orthogonal and symplectic involutions on these algebras are classified up to isomorphism by invariants of involutions in orthogonal group.

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1 Introduction

Let A be a central simple algebra over a field F. An anti-automorphism $\sigma : A \to A$ is called an *involution* if $\sigma^2 = \mathrm{id}$. Every nondegenerate bilinear form $B : V \times V \to F$ on a finite-dimensional F-vector space V induces a unique involution σ_B on $\mathrm{End}_F(V)$ which satisfies $B(x, f(y)) = B(\sigma_B(f)(x), y)$ for every $x, y \in V$ and $f \in \mathrm{End}_F(V)$. This involution is called the *adjoint involution* of $\mathrm{End}_F(V)$ with respect to B. The map $B \mapsto \sigma_B$ defines a one-to-one correspondence between the similarity classes of nondegenerate bilinear forms over F and the isomorphism classes of split F-algebras with involution (see [2, p. 1]).

Let (V, q) be a quadratic space over a field F. The group of all isometries of (V, q) is called the *orthogonal group* of (V, q) and is denoted by $O(V, q)$. An isometry $\tau \in O(V, q)$ is called an *involution* if $\tau^2 = \mathrm{id}$. Every involution $\tau \in O(V, q)$ induces a *natural* involution J_τ on the Clifford algebra $C(V, q)$ which satisfies $J_\tau(v) = v$ for every $v \in V$. The natural involutions were studied in [6] and [7] in connection with the Pfister Factor Conjecture, which was finally settled in [1]. Some properties of these involutions were also investigated in [3] and [5]. It is shown that for every multiquaternion algebra with involution $(A, \sigma) := (Q_1, \sigma_1) \otimes \cdots \otimes (Q_n, \sigma_n)$, there exists a quadratic space (V, q) and an involution $\tau \in O(V, q)$ such that $(A, \sigma) \simeq (C(V, q), J_\tau)$ (see [3, (6.3)] and [5, (6.3)]). This shows that properties of multiquaternion algebras with involution are reflected in properties of Clifford algebras with natural involution.

The main object of this work is to study the natural involutions of split Clifford algebras in characteristic 2. The transpose involution is the most elementary involution on the matrix algebra $M_n(F)$ over a field F. For a quadratic space (V, q) over a field F of characteristic 2, we obtain a necessary and sufficient condition to have $(C(V, q), J_\tau) \simeq (M_{2^n}(F), t)$. More generally, we characterize orthogonal and symplectic natural involutions on split Clifford algebras.

Following an approach based on the ideas of [3] and [5], we start with some observations on involutions of orthogonal group in characteristic 2. In [8, Theorem 1] it is shown that