Steiner triple systems with forbidden configurations

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Abstract

We discuss several characterizations of special classes of Steiner triple systems in terms of forbidden configurations. Among other things, we present such a characterization for strongly anti-Pasch Steiner triple systems.

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1 Introduction

Steiner triple systems are classical objects in combinatorial design theory. A *Steiner triple system* (STS for short) is a pair $S = (X, \mathcal{B})$ where X is a set of v points and $\mathcal{B}$ is a set of 3-subsets of X, called the triples of S, such that every two distinct points are contained in exactly one triple of S. One of the most classical results in combinatorics asserts that a Steiner triple system with v points exists if and only if $v \equiv 1, 3 \pmod 6$, $v \geq 3$. See [2] for a through treatment of enormous results on Steiner triple systems.

There are several prominent classes of Steiner triple systems of which we recall projective, affine and Hall STS in what follows. A *projective Steiner triple system* $\mathrm{PG}(d, 2)$ is the Steiner triple system with $2^{d+1} - 1$ points corresponding to non-zero $(d + 1)$-dimensional vectors over $\mathbb{Z}_2$ for $d \geq 1$. Three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ form a triple of $\mathrm{PG}(d, 2)$ if $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$. The smallest non-trivial projective Steiner triple system is $\mathrm{PG}(2, 2)$ which is indeed the Fano plane. An *affine Steiner triple system* $\mathrm{AG}(d, 3)$ is the Steiner triple system with 3^d points corresponding to d-dimensional vectors over $\mathbb{Z}_3$ for $d \geq 1$. Three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ form a triple of $\mathrm{AG}(d, 3)$ if $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$. The smallest non-trivial affine Steiner triple system, $\mathrm{AG}(2, 3)$, is the unique Steiner triple system with nine points which we denote it by S_9. Another interesting family of Steiner triple systems is the class of *Hall triple systems*. A Steiner triple system S is a Hall triple system if for every point x of S, there exists an involutory automorphism of S that fixes only the point x. Hall [5] showed that Hall triple systems are “locally” affine Steiner triple systems. To be more precise, a STS is a Hall STS if and only if every Steiner triple system induced by the points of two non-disjoint triples of S is isomorphic to S_9.

There are several characterizations for certain classes of combinatorial objects in terms of well-described forbidden substructures. For instance, the celebrated Kuratowski's theorem asserts that a graph is planar if and only if it does not contain a subdivision of one