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Generalized n-Laplacian: Quasilinear nonhomogenous problem with critical growth

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ABSTRACT

Applying the generalized Moser–Trudinger inequality, the Mountain Pass Theorem and the Ekeland Variational Principle we study the existence of non-trivial weak solutions to the problem

$$-\mathrm{div}\left(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)+V(x)\Phi'(|u|)\frac{u}{|u|}=f(x,u)+\mu h(x),$$
$$x\in\mathbb{R}^n,\ u\in W^1L^\Phi(\mathbb{R}^n)$$

where Φ is a Young function such that the space $W^1L^\Phi(\mathbb{R}^n)$ is embedded into exponential or multiple exponential Orlicz space, the nonlinearity $f(x,t)$ has the corresponding critical growth, $V(x)$ is a continuous potential, $h\in(L^\Phi(\mathbb{R}^n))^*$ is a nontrivial continuous function and $\mu>0$ is a small parameter.

1. Introduction

It is an often studied problem to find solutions to the Laplace equation

$$u\in W_0^{1,2}(\Omega)\quad\text{and}\quad -\Delta u=f(x,u)\quad\text{in }\Omega\subset\mathbb{R}^2.\tag{1.1}$$

For $n\geq 3$ and f satisfying $\lim_{t\to\infty}\frac{f(x,t)}{t^q}=0$ uniformly on Ω with $q<\frac{n+2}{n-2}$, there are many results using the compactness of the embedding of the space $W_0^{1,2}(\Omega)$ into $L^r(\Omega)$ with $r\in\left[1,\frac{2n}{n-2}\right)$ (see a review article by Lions [1] and the references given there). Problem (1.1) under condition $\lim_{t\to\infty}\frac{f(x,t)}{t^{\frac{n+2}{n-2}}}=0$ becomes much more difficult thanks to the fact that the embedding of $W_0^{1,2}(\Omega)$ into $L^{\frac{2n}{n-2}}(\Omega)$ is no longer compact. This difficulty has been overcame by Brezis and Nirenberg [2]. Their method uses the Mountain Pass Theorem by Ambrosetti and Rabinowitz [3].

When $n=2$, we do not only have the Sobolev embedding into $L^r(\Omega)$ for any $r\in[0,\infty)$ but there is also the Trudinger embedding [4] into the Orlicz space $\exp L^{\frac{n}{n-1}}(\Omega)$. In particular, there is the so-called Moser–Trudinger inequality by Moser [5]

$$\sup_{\|u\|_{W_0^{1,n}(\Omega)}\leq 1}\int_\Omega \exp\left(K|u|^{\frac{n}{n-1}}\right)\mathrm{d}x\leq C(n,\mathcal{L}_n(\Omega))\quad\text{if and only if}\quad K\leq n\omega_{n-1}^{\frac{1}{n-1}}.$$

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