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The projective vector field of a kind of three-dimensional quasi-homogeneous system on $\mathbb{S}^2$ ☆

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ABSTRACT

In this paper we study the projective vector field $\boldsymbol{Q}_T$ of a three-dimensional quasi-homogeneous system with weight $(1, 1, \alpha_3)$ and degree $\delta = 2, \alpha_3 \geq 2$. Projective vector fields $\boldsymbol{Q}_T$ of this kind are classified into two types. For one type, $\boldsymbol{Q}_T$ has no closed orbit and at most eight singularities, which lead to a global topology of the three-dimensional system. For the other type, $\boldsymbol{Q}_T$ has at most ten singularities. In addition, we show a relationship between $\boldsymbol{Q}_T$ and a Lienard system of this type. For both of them we obtain some conditions for the existence of limit cycles.

1. Introduction and statements of the main results

Let $\boldsymbol{Q}(\boldsymbol{x}) = (Q_1(\boldsymbol{x}), Q_2(\boldsymbol{x}), Q_3(\boldsymbol{x}))$ be a polynomial vector field in $\mathbb{R}^3$, where $\boldsymbol{x} = (x_1, x_2, x_3)$. We say that $\boldsymbol{Q}$ is *quasi-homogeneous with weight* $(\alpha_1, \alpha_2, \alpha_3)$ *and degree* δ if

$$\begin{aligned} Q_1(\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \lambda^{\alpha_3}x_3) &= \lambda^{\alpha_1-1+\delta}Q_1(x_1, x_2, x_3), \\ Q_2(\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \lambda^{\alpha_3}x_3) &= \lambda^{\alpha_2-1+\delta}Q_2(x_1, x_2, x_3), \\ Q_3(\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \lambda^{\alpha_3}x_3) &= \lambda^{\alpha_3-1+\delta}Q_3(x_1, x_2, x_3), \end{aligned} \tag{1}$$

where $\lambda \in \mathbb{R}$ and $\delta, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}^+$ (see [1,2]). The differential system

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{Q}(\boldsymbol{x}) \tag{2}$$

is called a *quasi-homogeneous polynomial system with weight* $(\alpha_1, \alpha_2, \alpha_3)$ *and degree* δ. In particular, (2) is a homogeneous polynomial system when $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$.

If we transform the coordinates such that

$$\boldsymbol{x} = (x_1, x_2, x_3) = (r^{\alpha_1}y_1, r^{\alpha_2}y_2, r^{\alpha_3}y_3), \quad \boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{S}^2, \ r \in \mathbb{R}^+,$$

then system (2) in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ turns into

$$\frac{dr}{dt} = \frac{r^{\delta}\langle \boldsymbol{y}, \boldsymbol{Q}(\boldsymbol{y})\rangle}{\langle \overline{\boldsymbol{y}}, \boldsymbol{y}\rangle}, \qquad \frac{d\boldsymbol{y}}{dt} = \frac{r^{\delta-1}(\langle \overline{\boldsymbol{y}}, \boldsymbol{y}\rangle \boldsymbol{Q}(\boldsymbol{y}) - \langle \boldsymbol{y}, \boldsymbol{Q}(\boldsymbol{y})\rangle \overline{\boldsymbol{y}})}{\langle \overline{\boldsymbol{y}}, \boldsymbol{y}\rangle},$$

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