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# A new topological degree theory for perturbations of the sum of two maximal monotone operators

Dhruba R. Adhikari[a], Athanassios G. Kartsatos[b,*]

[a] Department of Sciences and Mathematics, Mississippi University for Women, 1100 College Street, MUW-100, Columbus, MS 39701, USA

[b] Department of Mathematics, University of South Florida, Tampa, FL 33620-5700, USA

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ABSTRACT

Let $X$ be an infinite dimensional real reflexive Banach space with dual space $X^*$ and $G \subset X$, open and bounded. Assume that $X$ and $X^*$ are locally uniformly convex. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and strongly quasibounded, $S : X \supset D(S) \to X^*$ maximal monotone, and $C : X \supset D(C) \to X^*$ strongly quasibounded w.r.t. $S$ and such that it satisfies a generalized $(S_+)$-condition w.r.t. $S$. Assume that $D(S) = L \subset D(T) \cap D(C)$, where $L$ is a dense subspace of $X$, and $0 \in T(0), S(0) = 0$. A new topological degree theory is introduced for the sum $T+S+C$, with degree mapping $d(T+S+C, G, 0)$. The reason for this development is the creation of a useful tool for the study of a class of time-dependent problems involving three operators. This degree theory is based on a degree theory that was recently developed by Kartsatos and Skrypnik just for the single-valued sum $S+C$, as above.



## 1. Introduction—preliminaries

In what follows, the symbol $X$ stands for an infinite dimensional real reflexive Banach space, which has been renormed so that it and its dual $X^*$ are locally uniformly convex. The symbol $\|\cdot\|$ stands for the norm of $X, X^*$, and $J : X \to X^*$ is the normalized duality mapping.

In what follows, "continuous" means "strongly continuous" and the symbol "$\to$" ("$\rightharpoonup$") means strong (weak) convergence.

The symbol $\mathcal{R}$ ($\mathcal{R}_+$) stands for the set $(-\infty, \infty)$ ($[0, \infty)$) and the symbols $\partial D$, $\overline{D}$ denote the strong boundary and closure of the set $D$, respectively. We denote by $B_r(0)$ the open ball of $X$ or $X^*$ with center at zero and radius $r > 0$.

For an operator $T : X \to 2^{X^*}$, we denote by $D(T)$ the effective domain of $T$, i.e. $D(T) = \{x \in X : Tx \neq \emptyset\}$. We denote by $G(T)$ the graph of $T$, i.e. $G(T) = \{(x, y) : x \in D(T), y \in Tx\}$. An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "monotone", if for every $x, y \in D(T)$ and every $u \in Tx$, $v \in Ty$, we have

$$\langle u - v, x - y\rangle \geq 0.$$

A monotone operator $T$ is "maximal monotone" if $G(T)$ is maximal in $X \times X^*$, when $X \times X^*$ is partially ordered by inclusion. In our setting, a monotone operator $T$ is maximal if and only if $R(T + \lambda J) = X^*$ for all $\lambda \in (0, \infty)$. If $T$ is maximal monotone, then the operator $T_t \equiv (T^{-1} + tJ^{-1})^{-1} : X \to X^*$ is bounded, continuous (see Lemma 1), maximal monotone and such that $T_t x \rightharpoonup T^{\{0\}}x$ as $t \to 0^+$ for every $x \in D(T)$, where $T^{\{0\}}x$ denotes the element $y^* \in Tx$ of minimum norm, i.e. $\|T^{\{0\}}x\| = \inf\{\|y^*\| : y^* \in Tx\}$. In our setting, this infimum is always attained and $D(T^{\{0\}}) = D(T)$. Also, $T_t x \in TJ_t x$,

* Corresponding author. Tel.: +1 813 974 2643; fax: +1 813 974 2700.
*E-mail addresses:* dadhikari@as.muw.edu (D.R. Adhikari), hermes@math.usf.edu (A.G. Kartsatos).

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