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# Spectral properties of $p$-Laplacian problems with Neumann and mixed-type multi-point boundary conditions

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ABSTRACT

We consider the boundary value problem consisting of the $p$-Laplacian equation

$$-\phi_p(u')' = \lambda\phi_p(u), \quad \text{on } (-1, 1), \tag{1}$$

where $p > 1$, $\phi_p(s) := |s|^{p-1}\operatorname{sgn} s$ for $s \in \mathbb{R}$, $\lambda \in \mathbb{R}$, together with the multi-point boundary conditions

$$\phi_p(u'(\pm 1)) = \sum_{i=1}^{m^\pm} \alpha_i^\pm \phi_p(u'(\eta_i^\pm)), \tag{2}$$

or

$$u(\pm 1) = \sum_{i=1}^{m^\pm} \alpha_i^\pm u(\eta_i^\pm), \tag{3}$$

or a mixed pair of these conditions (with one condition holding at each of $x = -1$ and $x = 1$). In (2), (3), $m^\pm \geqslant 1$ are integers, $\eta_i^\pm \in (-1, 1)$, $1 \leqslant i \leqslant m^\pm$, and the coefficients $\alpha_i^\pm$ satisfy

$$\sum_{i=1}^{m^\pm} |\alpha_i^\pm| < 1.$$

We term the conditions (2) and (3), respectively, *Neumann-type* and *Dirichlet-type* boundary conditions, since they reduce to the standard Neumann and Dirichlet boundary conditions when $\alpha^\pm = 0$.

Given a suitable pair of boundary conditions, a number $\lambda$ is an *eigenvalue* of the corresponding boundary value problem if there exists a non-trivial solution $u$ (an *eigenfunction*). The *spectrum* of the problem is the set of eigenvalues. In this paper we obtain various spectral properties of these eigenvalue problems. We then use these properties to prove Rabinowitz-type, global bifurcation theorems for related bifurcation problems, and to obtain nonresonance conditions (in terms of the eigenvalues) for the solvability of related inhomogeneous problems.



## 1. Introduction

For any number $p > 1$, let $\phi_p(s) := |s|^{p-1}\operatorname{sgn} s$, $s \in \mathbb{R}$. For any integer $m \geqslant 1$, let $\mathcal{A}^m$ denote the set of $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ satisfying

$$\sum_{i=1}^{m} |\alpha_i| < 1. \tag{1.1}$$

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