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On Hartree equations with derivatives[☆]

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ABSTRACT

We consider the Cauchy problem of two types of Hartree equations with exchange–correlation correction terms:

$$\begin{cases} iu_t - \Delta u = V_k(u)u & \text{in } \mathbb{R}^{1+n}, \ k = 1, 2, \\ u(0) = \varphi & \text{in } \mathbb{R}^n, \ n \geq 1, \end{cases}$$

where

$$V_1(u) = |x|^{-\gamma} * (\lambda_1 |u|^2 + \lambda_2 |\nabla u|^2), \qquad V_2(u) = |x|^{-\gamma} * (\lambda ||\nabla|^{\delta} u|^2).$$

We establish the well-posedness of Cauchy problems and show the smoothing effect of solutions for each $0 < \gamma < n$ and $0 \leq \delta \leq 1$.

1. Introduction

In this paper, we consider two types of Schrödinger equations:

$$\begin{cases} iu_t + \Delta u = V_k(u)u & \text{in } \mathbb{R}^{1+n}, \\ u(0) = \varphi & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

$k = 1, 2$, where

$$V_1(u) = |x|^{-\gamma} * (\lambda_1 |u|^2 + \lambda_2 |\nabla u|^2), \qquad V_2(u) = |x|^{-\gamma} * (\lambda ||\nabla|^{\delta} u|^2),$$

$0 < \gamma < n$, $n \geq 1$, $0 \leq \delta \leq 1$ and $\lambda_1, \lambda_2, \lambda$ are nonzero real numbers. Here $|\nabla|^{\delta}$ denotes $(-\Delta)^{\frac{\delta}{2}}$. These equations are called as D-Hartree equations.

The equation with $\lambda_2 = 0$ or $\delta = 0$ and $\gamma = 1$ is called the Hartree one which shows the single-particle description of Coulombic many body systems in three space dimensions. In the density functional theory a more effective description has been studied for the system whose Hamiltonian $\mathcal{H}$ is defined by $-\Delta + V_{\rm ext} + V_{\rm H} + V_{\rm xc}$. Here we normalized the physical quantity $\hbar$, m by unity. $V_{\rm ext}$ is the external potential, $V_{\rm H}$ is the usual Hartree potential and $V_{\rm xc}$ is the exchange–correlation (xc) potential. If we consider dynamics of the system without an external potential, that is $V_{\rm ext} = 0$ and let $\psi = \psi(t, x)$ be the wavefunction of a single particle, then *heuristically* the function ψ satisfies that $i\psi_t = \mathcal{H}\psi$, where $V_{\rm H} = \tilde{\lambda} \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\, dy$, $\rho = |\psi|^2$ and $V_{\rm xc}$ is given by the formula

$$V_{\rm xc} = \int_{\mathbb{R}^3} \frac{\rho_{\rm xc}(x, y)}{|x - y|}\, dy$$

[☆] D-Hartree equations.

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