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Positive solutions to a semilinear elliptic equation with a Sobolev–Hardy term

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ABSTRACT

In this paper, we study the existence of positive solutions to the following semilinear elliptic equation with a Sobolev–Hardy term

$$\begin{cases} -\Delta u - \lambda u = \dfrac{u^{2^\sharp - 1}}{|y|} & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{0.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N (N \geq 3)$, $x = (y, z) \in \Omega \subset \mathbb{R}^k \times \mathbb{R}^{N-k} = \mathbb{R}^N$, $2 \leq k < N$, $2^\sharp := \frac{2(N-1)}{N-2}$ is the corresponding critical exponent and $0 < \lambda < \lambda_1$ where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. When $N \geq 4$, we prove that problem (0.1) has at least one positive solution by using the mountain-pass lemma and a global compactness result. The case $N = 3$ is quite different and we deal with this case by using the method in Jannelli (1999) [20] to prove the existence result. Moreover, we obtain the nonexistence result of (0.1) in a star shaped domain. Our main results extend a recent result of Castorina et al. (2009) [10] where $\lambda = 0$ and $\Omega = \mathbb{R}^N$.

1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$, with $2 \leq k < N$, $N \geq 3$. Suppose for some $(0, z_0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ such that $(0, z_0) \in \Omega$. Without loss of generality we suppose $0 \in \Omega$. Denote a point $x \in \mathbb{R}^N$ by $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$.

In this paper, we consider the following semilinear elliptic equation with the Sobolev–Hardy term

$$\begin{cases} -\Delta u - \lambda u = \dfrac{u^{2^\sharp - 1}}{|y|} & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where $2^\sharp := \frac{2(N-1)}{N-2}$ is the corresponding critical exponent, $0 < \lambda < \lambda_1$, λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. This problem is the Euler–Lagrange equation of the Hardy–Sobolev–Maz'ya inequality

$$C\left(\int_\Omega \frac{|u|^{2^\sharp}}{|y|} dydz\right)^{\frac{2}{2^\sharp}} \leq \int_\Omega [|\nabla u|^2 - \lambda u^2] dydz, \quad \forall u \in C_0^\infty(\Omega) \tag{1.2}$$

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