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Generalized gradients and characterization of epi-Lipschitz sets in Riemannian manifolds

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ABSTRACT

In this paper, a notion of generalized gradient on Riemannian manifolds is considered and a subdifferential calculus related to this subdifferential is presented. A characterization of the tangent cone to a nonempty subset S of a Riemannian manifold M at a point x is obtained. Then, these results are applied to characterize epi-Lipschitz subsets of complete Riemannian manifolds.

1. Introduction

Nondifferentiability appears naturally in several areas of mathematics and arises explicitly in the description of various modern technological systems. Nonsmooth analysis studies the local behavior of nondifferentiable functions and sets lacking smooth boundaries.

Nondifferentiable functions are often considered on finite dimensional or infinite dimensional Banach spaces, where the linear structure plays a central role. However, in various aspects of mathematics such as control theory and matrix analysis, nonsmooth functions arise naturally on smooth manifolds; see [1,2]. Unlike a Banach space, a manifold in general does not have a linear structure and therefore new techniques are needed for dealing with nonsmooth functions defined on manifolds. In the past few years, a number of results have been obtained on numerous aspects of nonsmooth analysis and their applications on Riemannian manifolds; see, e.g. [3–7].

Generalized gradients or subdifferentials refer to several set-valued replacements for the usual derivative. These concepts are used in developing differential calculus for nonsmooth functions. The concept of the generalized gradient of a locally Lipschitz function was introduced by Clarke in 1975. This concept reduces to the classical gradient for smooth functions and the subdifferential in the sense of convex analysis for convex functions and is accompanied by a useful calculus.

Attempts have been made to replace the class of locally Lipschitz functions by classes of noncontinuous functions and develop a subdifferential calculus; see, e.g. [6,8,9] and the references therein. For lower semicontinuous functions smooth local approximations from below led to the concept of viscosity and proximal subdifferentials.

In [3] the theory of viscosity solutions of Hamilton–Jacobi equations and the corresponding calculus were extended to the setting of Riemannian manifolds (possibly of infinite dimensional). In [4,10] a notion of proximal subdifferential for functions defined on Riemannian manifolds was introduced, a calculus for nonsmooth functions on these manifolds was established and its applications were discussed. By a different approach in [2] a nonsmooth calculus on finite dimensional Riemannian manifolds was developed and its applications to Hamilton–Jacobi equations were studied.

This paper is devoted to the study of the Clarke generalized gradient for locally Lipschitz functions defined on Riemannian manifolds (either finite or infinite dimensional). This notion was introduced in [3,11,12]. We develop a basic calculus result

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