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Weak geodesic flow on a semidirect product and global solutions to the periodic Hunter–Saxton system

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ABSTRACT

We give explicit solutions for the two-component Hunter–Saxton system on the unit circle. Moreover, we show how global weak solutions can be naturally constructed using the geometric interpretation of this system as a re-expression of the geodesic flow on the semidirect product of a suitable subgroup of the diffeomorphism group of the circle with the space of smooth functions on the circle. These spatially and temporally periodic solutions turn out to be conservative.

1. Introduction

In this paper, we are concerned with the two-component Hunter–Saxton system with periodic boundary conditions:

$$\begin{cases} u_{txx} + uu_{xxx} + 2u_x u_{xx} = \rho\rho_x, \quad t > 0,\ x \in \mathbb{S} \simeq \mathbb{R}/\mathbb{Z}, \\ \rho_t + (u\rho)_x = 0, \\ u(0, x) = \tilde{u}(x), \qquad \rho(0, x) = \tilde{\rho}(x). \end{cases} \tag{1.1}$$

The Hunter–Saxton system [1–5] is a two-component generalization of the well-known Hunter–Saxton equation $u_{txx} + uu_{xxx} + 2u_x u_{xx} = 0$ modeling the propagation of nonlinear orientation waves in a massive nematic liquid crystal (cf. [6–13]), to which it reduces if $\tilde{\rho}$ is chosen to vanish identically.

In mathematical physics, the Hunter–Saxton system (1.1) arises as a model for the nonlinear dynamics of one-dimensional non-dissipative dark matter (the so-called Gurevich–Zybin system; see [14] and the references therein). Additionally, it is the short wave limit (using the scaling $(t, x) \mapsto (\varepsilon t, \varepsilon x)$, and letting $\varepsilon \to 0$ in the resulting equations) of the two-component Camassa–Holm system originating in the Green–Naghdi equations which approximate the governing equations for water waves [15–18]. The Hunter–Saxton system is embedded in a more general family of coupled third-order systems [2] encompassing the axisymmetric Euler flow with swirl [19] and a vorticity model equation [20,21], among others (cf. [22–24]).

Geometric aspects of (1.1) have recently been described in [25]: the Hunter–Saxton system can be realized as a geodesic equation on the semidirect product of a subgroup of the group of circle diffeomorphisms with the space of smooth

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