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A proof of Hessian Sobolev inequality

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ABSTRACT

In this paper, taking the Hessian Sobolev inequality ($0 < p \leq k$) (X.-J. Wang, 1994 [2]) as the starting point, we give a proof of the Hessian Sobolev inequality when $k < p \leq k^*$, where k^* is the critical Sobolev embedding index of k-Hessian type. We also prove that k^* is optimal by one-dimensional Hardy's inequality.

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1. Introduction

First we introduce some standard notations and definitions, which can be found in [1,2].

Suppose Ω is a bounded smooth domain in $\mathbb{R}^n$, for a function $u \in C^2(\Omega)$, we define the k-Hessian operator $S_k(D^2u)$ by

$$S_k(D^2u) = S_k(\lambda(D^2u)), \tag{1.1}$$

where $1 \leq k \leq n$, $\lambda = (\lambda_1, \dots, \lambda_n)$ denotes the eigenvalues of the Hessian matrix of u, namely, D^2u. S_k is a kth elementary function on $\mathbb{R}^n$, given by

$$S_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{1.2}$$

Alternatively, $S_k(D^2u)$ can be written as the sum of the $k \times k$ principal minors of D^2u. When $k = 1$, $S_k(D^2u) = \Delta u$ is the Laplace operator, when $k = n$, $S_k(D^2u) = \det(D^2u)$ is the Monge–Ampere operator.

As is well known, the k-Hessian equation ($k \geq 2$) is a class of fully nonlinear partial differential equations of second order. To work with the elliptic realm, we introduce the class of k-admissible functions. Following [3], a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is called a k-admissible function if $\lambda(D^2u(x))$, $x \in \Omega$, belongs to the symmetric cone given by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_j(\lambda) > 0,\ j = 1, \dots, k\}. \tag{1.3}$$

The k-Hessian operator is elliptic at any k-admissible function u, i.e.

$$\{S_{ij}(D^2u)\} = \left\{\frac{\partial}{\partial r_{ij}} S_k(D^2u)\right\}$$

is positive definite.

We also need a geometric condition on the boundary $\partial\Omega$. A bounded domain Ω is strictly $(k-1)$-convex if $\partial\Omega$ satisfies

$$S_{k-1}(\kappa) \geq \delta_0 > 0 \tag{1.4}$$

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