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Sharp Nash inequalities on manifolds with boundary in the presence of symmetries

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ABSTRACT

In this paper we establish the best constant $\widetilde{A}_{opt}(\overline{M})$ for the trace Nash inequality on a n-dimensional compact Riemannian manifold in the presence of symmetries, which is an improvement over the classical case due to the symmetries which arise and reflect the geometry of manifold. This is particularly true when the data of the problem is invariant under the action of an arbitrary compact subgroup G of the isometry group $Is(M, g)$, where all the orbits have infinite cardinal.

1. Introduction

We say that the Nash inequality (1) is valid if there exists a constant $A > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$

$$\left(\int_{\mathbb{R}^n} u^2 \mathrm{d}x\right)^{1+\frac{2}{n}} \leqslant A \int_{\mathbb{R}^n} |\nabla u|^2 \,\mathrm{d}x \left(\int_{\mathbb{R}^n} |u| \,\mathrm{d}x\right)^{\frac{4}{n}}. \tag{1}$$

Such an inequality first appeared in the celebrated paper of Nash [1], where he discussed the Hölder regularity of solutions of divergence form in uniformly elliptic equations. It is a particular case of the Gagliardo–Nirenberg type inequalities $\|u\|_r \leqslant C\|\nabla u\|_q^a \|u\|_s^{1-a}$ and it is well known that the Nash inequality (1) and the Euclidean type Sobolev inequality are equivalent in the sense that if one of them is valid, the other one is also valid (i.e. see [2]). It is, also, well known that with this procedure of passing from the one type of inequalities to the other, is impossible to compare the best constants, since the inequalities under use are not optimal.

As far as the optimal version of Nash inequality (1) is concerned, the best constant $A_0(n)$, that is

$$A_0(n)^{-1} = \inf\left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \left(\int_{\mathbb{R}^n} |u| \mathrm{d}x\right)^{\frac{4}{n}}}{\left(\int_{\mathbb{R}^n} u^2 \mathrm{d}x\right)^{1+\frac{2}{n}}} \,\middle|\, u \in C_0^\infty\left(\mathbb{R}^n\right), u \not\equiv 0 \right\},$$

has been computed by Carlen and Loss in [3], together with the characterization of the extremals for the corresponding optimal inequality, as

$$A_0(n) = \frac{(n+2)^{\frac{n+2}{n}}}{2^{\frac{2}{n}} n \lambda_1^N |\mathcal{B}^n|^{\frac{2}{n}}},$$

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