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Regularized algorithms for hierarchical fixed-point problems

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ABSTRACT

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a non-expansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of non-expansive mappings. The purpose of this paper is to find the minimum norm solution of the following general hierarchical fixed point problem

$$\text{Find } \tilde{x} \in \bigcap_{n=1}^{\infty} Fix(T_n) \text{ such that } \langle \tilde{x} - S\tilde{x}, \tilde{x} - x \rangle \le 0, \quad \forall x \in \bigcap_{n=1}^{\infty} Fix(T_n).$$

We introduce an explicit regularized algorithm with strong convergence for finding the minimum norm solution of the above hierarchical fixed point problem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, let C be a nonempty closed convex subset of H. Let $S, T : C \to C$ be two non-expansive mappings. We use $Fix(T)$ to denote the set of fixed points of T. Now, we concern the following problem of finding hierarchically a fixed point of a non-expansive mapping T with respect to another mapping S, namely

$$\text{Find } \tilde{x} \in Fix(T) \text{ such that } \langle \tilde{x} - S\tilde{x}, \tilde{x} - x \rangle \le 0, \quad \forall x \in Fix(T). \tag{1.1}$$

Problem (1.1) is very important in the area of optimization and related fields, such as signal processing and image reconstruction (see [1–4]). Some algorithms for solving hierarchical fixed point problem (1.1) have been considered in the literature; see for example, [5–11] and the references therein. In many problems, it is needed to find a solution with minimum norm. A typical example is the least-squares solution to the constrained linear inverse problem; please see [12]. Therefore, it is an interesting problem to find the minimum-norm solution $\tilde{x}$ of (1.1). In this respect, very recently, Yao et al. [11] introduced an implicit algorithm and an explicit algorithm. Motivated and inspired by the works in this field, the purpose of this paper is dedicated to find the minimum norm solution of the following general hierarchical fixed point problem

$$\text{Find } \tilde{x} \in \bigcap_{n=1}^{\infty} Fix(T_n) \text{ such that } \langle \tilde{x} - S\tilde{x}, \tilde{x} - x \rangle \le 0, \quad \forall x \in \bigcap_{n=1}^{\infty} Fix(T_n), \tag{1.2}$$

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where $S : C \to C$ is a non-expansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ are an infinite family of non-expansive mappings. We first introduce an explicit regularized algorithm for finding the minimum norm solution of (1.2). Consequently, we show that the proposed algorithm has strong convergence under some mild assumptions.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $P : C \to H$ be a (possibly non-self) γ-contraction, where $\gamma \in [0, 1)$; namely,

$$\|Px - Py\| \le \gamma \|x - y\| \quad \text{for all } x, y \in C.$$

Recall that a mapping $T : C \to C$ is non-expansive if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all } x, y \in C.$$

The metric (or nearest point) projection from H onto C is the mapping $proj_C : H \to C$ which assigns to each point $x \in C$ the unique point $proj_C x \in C$ satisfying the property

$$\|x - proj_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Note that $proj_C$ is non-expansive and monotone. Below we gather some basic facts that are needed in the argument of the subsequent sections.

Lemma 2.1 (*[13] Demiclosedness Principle for Non-expansive Mappings*)**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a non-expansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in Fix(T)$.*

Lemma 2.2. *Given $x \in H$ and $z \in C$.*

(i) *That $z = proj_C x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \le 0 \quad \text{for all } y \in C.$$

(ii) *There holds the relation*

$$\langle proj_C x - proj_C y, x - y\rangle \ge \|proj_C x - proj_C y\|^2 \quad \text{for all } x, y \in H.$$

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be infinite family of non-expansive mappings and let $\{\xi_i\}_{i=1}^{\infty}$ be real number sequences such that $0 \le \xi_i \le 1$ for every $i \in \mathbf{N}$. For any $n \in \mathbf{N}$, define a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \xi_n T_n U_{n,n+1} + (1 - \xi_n)I, \\
U_{n,n-1} &= \xi_{n-1} T_{n-1} U_{n,n} + (1 - \xi_{n-1})I, \\
&\vdots \\
U_{n,k} &= \xi_k T_k U_{n,k+1} + (1 - \xi_k)I, \\
U_{n,k-1} &= \xi_{k-1} T_{k-1} U_{n,k} + (1 - \xi_{k-1})I, \\
&\vdots \\
U_{n,2} &= \xi_2 T_2 U_{n,3} + (1 - \xi_2)I, \\
W_n &= U_{n,1} = \xi_1 T_1 U_{n,2} + (1 - \xi_1)I.
\end{aligned} \tag{2.1}$$

Such W_n is called the W-mapping generated by $\{T_i\}_{i=1}^{\infty}$ and $\{\xi_i\}_{i=1}^{\infty}$.

We have the following crucial Lemma 2.3 concerning W_n which can be found in [14]. Now we only need the following similar version in Hilbert spaces.

Lemma 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be non-expansive mappings with $\bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. Let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \le b < 1$ for any $i \in \mathbf{N}$. Then we have the following results:*

(1) *for every $x \in C$ and $k \in \mathbf{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;*
(2) *$Fix(W) = \bigcap_{n=1}^{\infty} Fix(T_n)$.*

Lemma 2.4 (*See [15]*). *Using Lemma 2.3, one can define a mapping W of C into itself as: $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, for every $x \in C$. If $\{x_n\}$ is a bounded sequence in C, then we have*

$$\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0.$$

Lemma 2.5 (*[16]*). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we will introduce an explicit algorithm for finding the minimum norm solution of hierarchical fixed point problem (1.2). More precisely, we consider the following regularized algorithm

$$x_{n+1} = \alpha_n Sx_n + (1-\alpha_n)W_n proj_C[(1-\beta_n)x_n], \quad n \ge 0, \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real numbers in $(0, 1)$, $S : C \to C$ is a non-expansive mapping and $W_n : C \to C$ is the W-mapping defined by (2.1). Throughout, we use Ω to denote the set of solutions (1.2) and assume that Ω is nonempty.

Remark 3.1. We note that the well-known Mann algorithm $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$ has only weak convergence; please see [17–23] for the related works. This implies that the algorithm

$$x_{n+1} = \alpha_n Sx_n + (1-\alpha_n)W_n x_n, \quad n \ge 0 \tag{3.2}$$

has only weak convergence. In order to obtain strong convergence, some modifications are needed. We modify the algorithm (3.2) by adding the factor $1-\beta_n$ (where $\beta_n \to 0$). However, we note that $(1-\beta_n)x_n$ may not be in C. Hence, the projection $proj_C$ is used in order to guarantee that the sequence $\{x_n\}$ is well-defined.

Next, we will show the strong convergence of the algorithm (3.1). As a matter of fact, we introduce a general algorithm which includes the algorithm (3.1) as a special case.

Algorithm 3.2. For any given $x_0 \in C$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = \alpha_n Sx_n + (1-\alpha_n)W_n proj_C[\beta_n Px_n + (1-\beta_n)x_n], \quad n \ge 0, \tag{3.3}$$

where $P : C \to H$ is a γ-contraction.

It is clear if we take $P = 0$, then (3.3) reduces to (3.1). For the strong convergence of the algorithms (3.1) and (3.3), we have the following theorem.

Theorem 3.3. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a non-expansive mapping and $W_n : C \to C$ be the W-mapping defined by (2.1). Let $P : C \to H$ be a γ-contraction with $\gamma \in [0, 1)$. Suppose $\Omega \ne \emptyset$ and the following conditions are satisfied*

(C1) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \lim_{n\to\infty} \frac{1}{\beta_n}\left(\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right) = \lim_{n\to\infty} \frac{\beta_n - \beta_{n-1}}{\alpha_n\beta_n} = \lim_{n\to\infty} \frac{1}{\alpha_n\beta_n}\prod_{i=1}^{n-1}\xi_i = 0$;
(C2) $\sum_{n=0}^{\infty} \beta_n = \infty$.

Let the sequence $\{x_n\}$ be defined by (3.3). Then, we have

(i) $\lim_{n\to\infty} \frac{\|x_{n+1}-x_n\|}{\alpha_n} = 0$;
(ii) *every weak cluster point of $\{x_n\}$ solves the following variational inequality*

$$\tilde{x} \in \Omega, \quad \langle (I-P)\tilde{x}, x-\tilde{x}\rangle \ge 0, \ \forall x \in \Omega. \tag{3.4}$$

Further, if we add the following additional assumptions

(a1) $\lim_{n\to\infty} \frac{\alpha_n^2}{\beta_n} = 0$;
(a2) *There exists some constant $k > 0$ such that $\|x - W_n x\| \ge kDist(x, \bigcap_{n=1}^{\infty} Fix(T_n))$, where $Dist(x, \bigcap_{n=1}^{\infty} Fix(T_n)) = \inf_{y\in\bigcap_{n=1}^{\infty} Fix(T_n)} \|x-y\|$.*

Then the sequence $\{x_n\}$ generated by (3.3) converges strongly to $\tilde{x} \in \bigcap_{n=1}^{\infty} Fix(T_n)$ which solves variational inequality (3.4). In particular, if we take $P = 0$, then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $\tilde{x} \in \bigcap_{n=1}^{\infty} Fix(T_n)$ which is the minimum norm solution of hierarchical fixed point problem (1.2).

Remark 3.4. We can choose the following parameters satisfying conditions (C1), (C2) and (a1), for instance,

$$\alpha_n = \frac{1}{n^{1/4}}, \qquad \beta_n = \frac{1}{n^{1/3}}, \qquad \xi_n = \frac{1}{2}, \quad n \geq 1.$$

We have the following important remark (see [6]).

Remark 3.5. (1) The hypothesis (a2) was used in [24] by Senter and Dotson so as to obtain a strong convergence result for Mann iterates. Later Maiti and Ghosh [25], Tan and Xu [26] studied the approximation of fixed-points of a non-expansive mapping T by Ishikawa iterates under the condition introduced in [16] and pointed out that this assumption is weaker than the requirement that the mapping T is demi-compact.

(2) Since any weak-cluster point of $\{x_n\}$ is in $\bigcap_{n=1}^{\infty} Fix(T_n)$ (see the detailed proof below), we would like to emphasize that it is enough to assume that (a2) holds true in a neighborhood of $\bigcap_{n=1}^{\infty} Fix(T_n)$.

(3) We would also like to note that, thanks to a result by Lemaire [27], (a2) is in the convex minimization setting equivalent to

$$\forall x \in H, \quad \varphi(x) - \min \varphi \geq kDist(x, \arg\min \varphi)^{1/2}$$

which is exactly one of the assumptions used in [28] to obtain convergence results (Propositions 3.4 and 4.3) of a proximal method for hierarchical minimization problems. In [28], the convergence results are valid only in the finite dimensional case.

Now we divide our detailed proof into several conclusions. Next, we assume that all conditions of Theorem 3.3 are satisfied.

Conclusion 3.6. (a) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;
(b) $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$.

Proof. Setting $y_n = \beta_n Px_n + (1-\beta_n)x_n$ for all $n \geq 0$. It follows that

$$\begin{aligned} y_n - y_{n-1} &= \beta_n Px_n + (1-\beta_n)x_n - \beta_{n-1}Px_{n-1} - (1-\beta_{n-1})x_{n-1} \\ &= \beta_n(Px_n - Px_{n-1}) + (\beta_n - \beta_{n-1})Px_{n-1} + (1-\beta_n)(x_n - x_{n-1}) + (\beta_{n-1} - \beta_n)x_{n-1}. \end{aligned}$$

It follows that

$$\begin{aligned} \|y_n - y_{n-1}\| &\leq \gamma\beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Px_{n-1}\| + \|x_{n-1}\|) \\ &= [1 - (1-\gamma)\beta_n]\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Px_{n-1}\| + \|x_{n-1}\|). \end{aligned} \tag{3.5}$$

From (3.3), we have

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n Sx_n + (1-\alpha_n)W_n proj_C[y_n] - \alpha_{n-1}Sx_{n-1} - (1-\alpha_{n-1})W_{n-1}proj_C[y_{n-1}] \\ &= \alpha_n(Sx_n - Sx_{n-1}) + (1-\alpha_n)\left(W_n proj_C[y_n] - W_n proj_C[y_{n-1}]\right) \\ &\quad + (1-\alpha_n)\Big(W_n proj_C[y_{n-1}] - W_{n-1}proj_C[y_{n-1}]\Big) + (\alpha_n - \alpha_{n-1})Sx_{n-1} + (\alpha_{n-1} - \alpha_n)W_{n-1}proj_C[y_{n-1}]. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n\|Sx_n - Sx_{n-1}\| + (1-\alpha_n)\|W_n proj_C[y_n] - W_n proj_C[y_{n-1}]\| \\ &\quad + (1-\alpha_n)\|W_n proj_C[y_{n-1}] - W_{n-1}proj_C[y_{n-1}]\| + |\alpha_n - \alpha_{n-1}|(\|Sx_{n-1}\| + \|W_{n-1}proj_C[y_{n-1}]\|) \\ &\leq \alpha_n\|x_n - x_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| + (1-\alpha_n)\|W_n proj_C[y_{n-1}] - W_{n-1}proj_C[y_{n-1}]\| \\ &\quad + |\alpha_n - \alpha_{n-1}|(\|Sx_{n-1}\| + \|W_{n-1}proj_C[y_{n-1}]\|). \end{aligned} \tag{3.6}$$

From (2.1), since T_i and $U_{n,i}$ are non-expansive, we have

$$\begin{aligned} \|W_n proj_C[y_{n-1}] - W_{n-1}proj_C[y_{n-1}]\| &= \|\xi_1 T_1 U_{n,2} proj_C[y_{n-1}] - \xi_1 T_1 U_{n-1,2} proj_C[y_{n-1}]\| \\ &\leq \xi_1\|U_{n,2}proj_C[y_{n-1}] - U_{n-1,2}proj_C[y_{n-1}]\| \\ &= \xi_1\|\xi_2 T_2 U_{n,3} proj_C[y_{n-1}] - \xi_2 T_2 U_{n-1,3} proj_C[y_{n-1}]\| \\ &\leq \xi_1\xi_2\|U_{n,3}proj_C[y_{n-1}] - U_{n-1,3}proj_C[y_{n-1}]\| \\ &\leq \cdots \\ &\leq \xi_1\xi_2\cdots\xi_{n-1}\|U_{n,n}proj_C[y_{n-1}] - U_{n-1,n}proj_C[y_{n-1}]\| \\ &\leq M\prod_{i=1}^{n-1}\xi_i, \end{aligned} \tag{3.7}$$

where M is some constant such that

$$\sup_n \left\{ \|U_{n,n} proj_C[y_{n-1}] - U_{n-1,n} proj_C[y_{n-1}]\|, \|x_n - x_{n-1}\|, (\|Sx_{n-1}\| + \|W_{n-1} proj_c[y_{n-1}]\|), (\|Px_{n-1}\| + \|x_{n-1}\|) \right\} \le M.$$

Substituting (3.5) and (3.7) into (3.6), we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\le \alpha_n \|x_n - x_{n-1}\| + (1-\alpha_n)[1 - (1-\gamma)\beta_n]\|x_n - x_{n-1}\| + M \prod_{i=1}^{n-1} \xi_i \\ &\quad + |\beta_n - \beta_{n-1}|(\|Px_{n-1}\| + \|x_{n-1}\|) + |\alpha_n - \alpha_{n-1}|(\|Sx_{n-1}\| + \|W_{n-1} proj_C[y_{n-1}]\|) \\ &= [1 - (1-\gamma)\beta_n(1-\alpha_n)]\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Px_{n-1}\| + \|x_{n-1}\|) \\ &\quad + M \prod_{i=1}^{n-1} \xi_i + |\alpha_n - \alpha_{n-1}|(\|Sx_{n-1}\| + \|W_{n-1} proj_C[y_{n-1}]\|). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\|x_{n+1} - x_n\|}{\alpha_n} &\le [1 - (1-\gamma)\beta_n(1-\alpha_n)] \frac{\|x_n - x_{n-1}\|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n}(\|Px_{n-1}\| + \|x_{n-1}\|) \\ &\quad + \frac{M}{\alpha_n} \prod_{i=1}^{n-1} \xi_i + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n}(\|Sx_{n-1}\| + \|W_{n-1} proj_C[y_{n-1}]\|) \\ &= [1 - (1-\gamma)\beta_n(1-\alpha_n)] \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} + \frac{M}{\alpha_n} \prod_{i=1}^{n-1} \xi_i + [1 - (1-\gamma)\beta_n(1-\alpha_n)] \\ &\quad \times \left(\frac{\|x_n - x_{n-1}\|}{\alpha_n} - \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \right) + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n}(\|Px_{n-1}\| + \|x_{n-1}\|) \\ &\quad + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n}(\|Sx_{n-1}\| + \|W_{n-1} proj_C[y_{n-1}]\|) \\ &\le [1 - (1-\gamma)\beta_n(1-\alpha_n)] \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\ &\quad + \left(\left| \frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}} \right| + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} + \frac{1}{\alpha_n} \prod_{i=1}^{n-1} \xi_i \right) M \\ &= [1 - (1-\gamma)\beta_n(1-\alpha_n)] \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} + (1-\gamma)\beta_n(1-\alpha_n) \frac{M}{(1-\gamma)(1-\alpha_n)} \\ &\quad \times \left(\frac{1}{\beta_n} \left| \frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}} \right| + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n \beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n \beta_n} + \frac{1}{\alpha_n \beta_n} \prod_{i=1}^{n-1} \xi_i \right). \end{aligned} \tag{3.8}$$

From (C1), we note that $\lim_{n\to\infty} \frac{1}{\alpha_{n-1}} \left(\frac{\alpha_n - \alpha_{n-1}}{\beta_n \alpha_n} \right) = 0$ which implies that

$$\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n \beta_n} = 0. \tag{3.9}$$

Thus, from (C1) and (3.9), we have

$$\lim_{n\to\infty} \left(\frac{1}{\beta_n} \left| \frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}} \right| + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n \beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n \beta_n} + \frac{1}{\alpha_n \beta_n} \prod_{i=1}^{n-1} \xi_i \right) = 0.$$

Hence, applying Lemma 2.5 to (3.8), we immediately conclude that

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\alpha_n} = 0.$$

This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

From (3.3) and (3.10), we have

$$\lim_{n\to\infty} \|x_n - W_n proj_C[y_n]\| = 0. \tag{3.11}$$

By (3.3), we get

$$\begin{aligned}\|proj_C[y_n] - x_n\| &= \|proj_C[y_n] - proj_C[x_n]\| \\ &\le \|y_n - x_n\| \\ &= \|\beta_n(Px_n - x_n)\| \to 0.\end{aligned} \tag{3.12}$$

Notice that

$$\begin{aligned}\|x_n - Wx_n\| &\le \|x_n - W_n proj_C[y_n]\| + \|W_n proj_C[y_n] - W_n x_n\| + \|W_n x_n - Wx_n\| \\ &\le \|x_n - W_n proj_C[y_n]\| + \|proj_C[y_n] - x_n\| + \|W_n x_n - Wx_n\|.\end{aligned} \tag{3.13}$$

By (3.11)–(3.13) and Lemma 2.4, we deduce

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \quad \square \tag{3.14}$$

Conclusion 3.7. $\omega_w(x_n) \subset Fix(W) = \bigcap_{n=1}^{\infty} Fix(T_n)$ and $\omega_w(x_n) \subset \Omega$.

Proof. Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $\tilde{x} \in H$. Therefore, $\tilde{x} \in Fix(W) = \bigcap_{n=1}^{\infty} Fix(T_n)$ by (3.14) and Lemma 2.1 (demi-closed principle). Hence, $\omega_w(x_n) \subset Fix(W) = \bigcap_{n=1}^{\infty} Fix(T_n)$. Next, we show that $\omega_w(x_n) \subset \Omega$.

Rewriting (3.3) as

$$x_{n+1} - x_n = \alpha_n(Sx_n - x_n) + (1-\alpha_n)(W_n proj_C[y_n] - proj_C[y_n]) + (1-\alpha_n)(Proj_C[y_n] - y_n) + (1-\alpha_n)(y_n - x_n),$$

that is

$$\frac{x_n - x_{n+1}}{\alpha_n} = (I-S)x_n + \frac{1-\alpha_n}{\alpha_n}(I-W_n)proj_C[y_n] + \frac{1-\alpha_n}{\alpha_n}(I-proj_C)y_n + \frac{\beta_n(1-\alpha_n)}{\alpha_n}(I-P)x_n.$$

Set $z_n = \frac{x_n - x_{n+1}}{\alpha_n}$ for all $n \ge 1$, that is,

$$z_n = (I-S)x_n + \frac{1-\alpha_n}{\alpha_n}(I-W_n)proj_C[y_n] + \frac{1-\alpha_n}{\alpha_n}(I-proj_C)y_n + \frac{\beta_n(1-\alpha_n)}{\alpha_n}(I-P)x_n.$$

Pick up $u \in \bigcap_{n=1}^{\infty} Fix(T_n)$. Then, we have

$$\begin{aligned}\langle z_n, x_n - u\rangle &= \langle (I-S)x_n - (I-S)u, x_n - u\rangle + \langle (I-S)u, x_n - u\rangle \\ &\quad + \frac{1-\alpha_n}{\alpha_n}\langle (I-W_n)proj_C[y_n] - (I-W_n)u, proj_C[y_n] - u\rangle \\ &\quad + \frac{1-\alpha_n}{\alpha_n}\langle (I-W_n)proj_C[y_n], x_n - proj_C[y_n]\rangle + \frac{1-\alpha_n}{\alpha_n}\langle (I-proj_C)y_n, x_n - proj_C[y_n]\rangle \\ &\quad + \frac{1-\alpha_n}{\alpha_n}\langle (I-proj_C)y_n, proj_C[y_n] - u\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-P)x_n, x_n - u\rangle.\end{aligned}$$

Using monotonicity of $I - W_n$ and $I - S$, we derive that

$$\langle (I-W_n)proj_C[y_n] - (I-W_n)u, proj_C[y_n] - u\rangle \ge 0 \quad \text{and} \quad \langle (I-S)x_n - (I-S)u, x_n - u\rangle \ge 0.$$

Using the property of the projection (Lemma 2.2), we have

$$\langle (I-proj_C)y_n, proj_C[y_n] - u\rangle \ge 0.$$

At the same time, we observe that

$$\begin{aligned}y_n - W_n proj_C[y_n] &= \beta_n Px_n + (1-\beta_n)x_n - W_n proj_C[y_n] \\ &= \beta_n(Px_n - x_{n+1}) + (1-\beta_n)(x_n - x_{n+1}) + x_{n+1} - W_n proj_C[y_n] \\ &= \beta_n(Px_n - x_{n+1}) + (1-\beta_n)(x_n - x_{n+1}) + \alpha_n(Sx_n - W_n proj_C[y_n]).\end{aligned}$$

Therefore,

$$\begin{aligned}\langle z_n, x_n-u\rangle &\geq \langle (I-S)u, x_n-u\rangle + \frac{1-\alpha_n}{\alpha_n}\langle (I-W_n)proj_C[y_n], x_n-proj_C[y_n]\rangle \\ &\quad + \frac{1-\alpha_n}{\alpha_n}\langle (I-proj_C)y_n, x_n-proj_C[y_n]\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-P)x_n, x_n-u\rangle \\ &= \langle (I-S)u, x_n-u\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-P)x_n, x_n-u\rangle + \frac{1-\alpha_n}{\alpha_n}\langle y_n-W_nproj_C[y_n], x_n-proj_C[y_n]\rangle \\ &= \langle (I-S)u, x_n-u\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-P)x_n, x_n-u\rangle + \frac{(1-\alpha_n)\beta_n}{\alpha_n}\langle Px_n-x_{n+1}, x_n-proj_C[y_n]\rangle \\ &\quad + (1-\alpha_n)(1-\beta_n)\left\langle \frac{x_n-x_{n+1}}{\alpha_n}, x_n-proj_C[y_n]\right\rangle + (1-\alpha_n)\langle Sx_n-W_nproj_C[y_n], x_n-proj_C[y_n]\rangle.\end{aligned}$$

But, since $z_n\to 0$, $\frac{\beta_n}{\alpha_n}\to 0$, $\frac{x_n-x_{n+1}}{\alpha_n}\to 0$ and $x_n-proj_C[y_n]\to 0$, we obtain from the above inequality that

$$\limsup_{n\to\infty}\langle (I-S)u, x_n-u\rangle \leq 0, \quad u\in\bigcap_{n=1}^{\infty}Fix(T_n).$$

Therefore,

$$\limsup_{j\to\infty}\langle (I-S)u, x_{n_j}-u\rangle \leq 0, \quad u\in\bigcap_{n=1}^{\infty}Fix(T_n).$$

Since $x_{n_j}\rightharpoonup\tilde{x}$, we have

$$\limsup_{j\to\infty}\langle (I-S)u, x_{n_j}-u\rangle = \langle (I-S)u, \tilde{x}-u\rangle.$$

This implies that every weak cluster point $\tilde{x}\in\bigcap_{n=1}^{\infty}Fix(T_n)$ of the sequence $\{x_n\}$ solves the variational inequality

$$\langle (I-S)u, \tilde{x}-u\rangle \leq 0, \quad u\in\bigcap_{n=1}^{\infty}Fix(T_n),$$

which is equivalent to its dual variational inequality (see [10])

$$\langle (I-S)\tilde{x}, \tilde{x}-u\rangle \leq 0, \quad u\in\bigcap_{n=1}^{\infty}Fix(T_n).$$

This suffices to guarantee that $\omega_w(x_n)\subset\Omega$. □

Conclusion 3.8. $\limsup_{n\to\infty}\langle P\tilde{x}-\tilde{x}, y_n-\tilde{x}\rangle \leq 0$ and $\limsup_{n\to\infty}\frac{\alpha_n}{\beta_n}\langle S\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle \leq 0$.

Proof. Note that P is a contraction. Then the solution set of the variational inequality (3.4) is a singleton. Let $\tilde{x}$ be the unique solution of the variational inequality (3.4). Now take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ satisfying

$$\limsup_{n\to\infty}\langle (I-P)\tilde{x}, x_n-\tilde{x}\rangle = \lim_{i\to\infty}\langle (I-P)\tilde{x}, x_{n_i}-\tilde{x}\rangle.$$

Without loss of generality, we may further assume that $x_{n_i}\rightharpoonup\bar{x}$, then $\bar{x}\in\Omega$. Therefore, noting that $\tilde{x}$ is the solution of the variational inequality (3.4), we get

$$\limsup_{n\to\infty}\langle (I-P)\tilde{x}, x_n-\tilde{x}\rangle = \langle (I-P)\tilde{x}, \bar{x}-\tilde{x}\rangle \geq 0.$$

We note that every weak cluster point of $\{x_n\}$ is in Ω. Since $y_n-x_n\to 0$, then every weak cluster point of $\{y_n\}$ is also in Ω. Consequently, since $\tilde{x}=proj_{\Omega}(P\tilde{x})$, we easily deduce that

$$\limsup_{n\to\infty}\langle P\tilde{x}-\tilde{x}, y_n-\tilde{x}\rangle \leq 0.$$

On the other hand, we note that

$$\langle S\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle = \langle S\tilde{x}-\tilde{x}, proj_{\bigcap_{n=1}^{\infty}Fix(T_n)}x_{n+1}-\tilde{x}\rangle + \langle S\tilde{x}-\tilde{x}, x_{n+1}-proj_{\bigcap_{n=1}^{\infty}Fix(T_n)}x_{n+1}\rangle.$$

Since $proj_{\bigcap_{n=1}^{\infty} Fix(T_n)} x_{n+1} \in \bigcap_{n=1}^{\infty} Fix(T_n)$, by (1.2) we have

$$\langle S\tilde{x} - \tilde{x}, proj_{\bigcap\limits_{n=1}^{\infty} Fix(T_n)} x_{n+1} - \tilde{x} \rangle \le 0,$$

and therefore,

$$\begin{aligned}
\langle S\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle &\le \langle S\tilde{x} - \tilde{x}, x_{n+1} - proj_{\bigcap\limits_{n=1}^{\infty} Fix(T_n)} x_{n+1} \rangle \\
&\le \|S\tilde{x} - \tilde{x}\| \, \|x_{n+1} - proj_{\bigcap\limits_{n=1}^{\infty} Fix(T_n)} x_{n+1}\| \\
&= \|S\tilde{x} - \tilde{x}\| \times Dist\left(x_{n+1}, \bigcap_{n=1}^{\infty} Fix(T_n)\right) \\
&\le \frac{1}{k} \|S\tilde{x} - \tilde{x}\| \, \|x_{n+1} - W_n x_{n+1}\|.
\end{aligned}$$

We note that

$$\begin{aligned}
\|x_{n+1} - W_n x_{n+1}\| &\le \|x_{n+1} - W_n proj_C[y_n]\| + \|W_n proj_C[y_n] - W_n x_n\| + \|W_n x_n - W_n x_{n+1}\| \\
&\le \alpha_n \|Sx_n - W_n proj_C[y_n]\| + \|y_n - x_n\| + \|x_{n+1} - x_n\| \\
&\le \alpha_n \|Sx_n - W_n proj_C[y_n]\| + \beta_n \|Px_n - x_n\| + \|x_{n+1} - x_n\|.
\end{aligned}$$

Hence, we have

$$\frac{\alpha_n}{\beta_n} \langle S\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \le \frac{\alpha_n^2}{k\beta_n} \|S\tilde{x} - \tilde{x}\| \, \|Sx_n - W_n proj_C[y_n]\| + \frac{\alpha_n}{k} \|S\tilde{x} - \tilde{x}\| \, \|Px_n - x_n\| + \frac{\alpha_n^2}{k\beta_n} \frac{\|x_{n+1} - x_n\|}{\alpha_n} \|S\tilde{x} - \tilde{x}\|.$$

It follows that

$$\limsup_{n\to\infty} \frac{\alpha_n}{\beta_n} \langle S\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \le 0. \quad \square$$

By Conclusions 3.6–3.8, we finally prove Theorem 3.3.

Proof. From (3.3), we have

$$x_{n+1} - \tilde{x} = \alpha_n (Sx_n - S\tilde{x}) + (1 - \alpha_n)(W_n proj_C[y_n] - \tilde{x}) + \alpha_n (S\tilde{x} - \tilde{x}).$$

Thus, we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\le \|\alpha_n (Sx_n - S\tilde{x}) + (1 - \alpha_n)(W_n proj_C[y_n] - \tilde{x})\|^2 + 2\alpha_n \langle S\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&\le \alpha_n \|Sx_n - S\tilde{x}\|^2 + (1 - \alpha_n) \|W_n proj_C[y_n] - \tilde{x}\|^2 + 2\alpha_n \langle S\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&\le \alpha_n \|x_n - \tilde{x}\|^2 + (1 - \alpha_n) \|y_n - \tilde{x}\|^2 + 2\alpha_n \langle S\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle.
\end{aligned} \tag{3.15}$$

At the same time, we observe that

$$\begin{aligned}
\|y_n - \tilde{x}\|^2 &= \|(1 - \beta_n)(x_n - \tilde{x}) + \beta_n (Px_n - P\tilde{x}) + \beta_n (P\tilde{x} - \tilde{x})\|^2 \\
&\le \|(1 - \beta_n)(x_n - \tilde{x}) + \beta_n (Px_n - P\tilde{x})\|^2 + 2\beta_n \langle P\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle \\
&\le (1 - \beta_n) \|x_n - \tilde{x}\|^2 + \beta_n \|Px_n - P\tilde{x}\|^2 + 2\beta_n \langle P\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle \\
&\le (1 - \beta_n) \|x_n - \tilde{x}\|^2 + \beta_n \gamma^2 \|x_n - \tilde{x}\|^2 + 2\beta_n \langle P\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle \\
&= [1 - (1 - \gamma^2)\beta_n] \|x_n - \tilde{x}\|^2 + 2\beta_n \langle P\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle.
\end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.15), we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\le \alpha_n \|x_n - \tilde{x}\|^2 + (1 - \alpha_n)[1 - (1 - \gamma^2)\beta_n] \|x_n - \tilde{x}\|^2 \\
&\quad + 2\beta_n (1 - \alpha_n) \langle P\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle + 2\alpha_n \langle S\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&= [1 - (1 - \gamma^2)\beta_n (1 - \alpha_n)] \|x_n - \tilde{x}\|^2 + 2\beta_n (1 - \alpha_n) \langle P\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle + 2\alpha_n \langle S\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&= [1 - (1 - \gamma^2)\beta_n (1 - \alpha_n)] \|x_n - \tilde{x}\|^2 + (1 - \gamma^2)\beta_n (1 - \alpha_n) \\
&\quad \times \left\{ \frac{2}{1 - \gamma^2} \langle P\tilde{x} - \tilde{x}, y_n - \tilde{x} \rangle + \frac{2}{(1 - \gamma^2)(1 - \alpha_n)} \frac{\alpha_n}{\beta_n} \langle S\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \right\}.
\end{aligned} \tag{3.17}$$

Therefore, we can apply Lemma 2.5 to (3.17) to conclude that $x_n \to \tilde{x}$. This completes the proof. $\square$

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