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Inequalities in metric spaces with applications

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ABSTRACT

Analogues of the parallelogram identity and the (CN) inequality of Bruhat and Tits in uniformly convex metric spaces are established. As an application of the new inequalities, we prove two fixed point results for single-valued and multi-valued Lipschitzian mappings.

1. Introduction

The Hilbert parallelogram identity is the following:

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$$

for any x, y in a Hilbert space H, and it plays a major role in proving many basic results. This identity implies

$$\|\lambda x+(1-\lambda)y\|^2+\lambda(1-\lambda)\|x-y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2$$

for any x, y in H and for any $\lambda \in [0,1]$. In [1], Xu gave a nice extension of these identities to uniformly convex Banach spaces. His work has been used as an important tool in proving many interesting results. In order to extend Xu's ideas to metric spaces, Beg [2] had to change the definition of uniform convexity in metric spaces. One of the difficulties in carrying out such extensions lies in the heavy use of the linear structure of the Banach spaces.

In this paper, we use the classical definition of uniform convexity in metric spaces and obtain an analogue of the parallelogram inequality and the (CN) inequality of Bruhat and Tits [3] in these spaces. Then we give several applications of our paper as in [1]. To the best of our knowledge this is the first attempt that successfully carries out such an extension on a nonlinear domain.

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