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Transitivity and variational principles

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ABSTRACT

We apply an order reasoning to mappings satisfying the triangle inequality. This general approach yields the Ekeland's variational principle as one of the consequences. In addition we obtain an extension of the Brøndsted variational principle and of the Takahashi fixed point theorem.

The variational principles require a kind of order. One can first assume that a set is equipped with an "order" relation and then we demand the mapping(s) under consideration to fulfil some additional conditions (see e.g., [1–4]). Another way is to define an "order" with the help of special mappings on a space equipped with some topological features. Such orders usually look like

$$x \le y \quad \text{iff } \psi(y) + d(y, x) - \psi(x) \le 0;$$

see, for instance, [5–8,4]. Our idea is to use a mapping $\varphi: X \times X \to R$ satisfying the triangle inequality which as well "orders" X and defines a kind of convergence in X.

Let X be a nonempty set and $\varphi: X \times X \to R$ a mapping satisfying:

$$\varphi(z, x) \le \varphi(z, y) + \varphi(y, x), \quad x, y, z \in X. \tag{1}$$

Then from $\varphi(x, x) \le 2\varphi(x, x)$ and (1) we obtain

$$0 \le \varphi(x, x) \le \varphi(x, y) + \varphi(y, x), \quad x, y \in X. \tag{2}$$

The subsequent three definitions describe the "world" of the present paper.

Definition 1. Let X be a nonempty set and $\varphi: X \times X \to R$ a mapping satisfying (1). Then

(i) $\mathcal{S} = (X, \varphi, x_0)$ is a local structure if $\varphi(\cdot, x_0)$ has a finite lower bound,
(ii) $\mathcal{S} = (X, \varphi)$ is a global structure if $\varphi(\cdot, x_0)$ has a finite lower bound for each $x_0 \in X$.

By a **structure** we will mean – according to the context – either of these.

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