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A counterexample to uniqueness of generalized characteristics in Hamilton–Jacobi theory

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ABSTRACT

The notion of generalized characteristics plays a pivotal role in the study of propagation of singularities for Hamilton–Jacobi equations. This note gives an example of nonuniqueness of forward generalized characteristics emanating from a given point.

1. Introduction

We are in this note concerned with generalized characteristics for the Hamilton–Jacobi equation

$$S_t + H(x, \nabla S) = 0 \quad \text{in } Q = (0, \infty) \times \mathbb{R}^n, \qquad S(0, x) = S_0(x) \quad \text{in } \mathbb{R}^n, \tag{1}$$

in the multidimensional case $n \geq 2$. While the existence of generalized characteristics is well-known, the corresponding uniqueness problem is largely unsettled. The purpose of the present contribution is to manifest that forward generalized characteristics are nonunique, in general. The Hamiltonian H appearing in (1) is the Legendre–Fenchel transform of a Lagrangian L. We assume the following conditions linking (1) to a problem in the calculus of variations.

(A) The Lagrangian L is from $C^2(\mathbb{R}^n \times \mathbb{R}^n)$. It fulfills $\nabla_v^2 L(x, v) > 0$ and $L(x, v) \geq \ell(|v|)$ for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ where $\ell(s)/s \to \infty$ as $s \to \infty$. The Hamiltonian H is given by

$$H(x, p) = \max_{v \in \mathbb{R}^n} (\langle p, v \rangle - L(x, v)), \quad (x, p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

(B) The initial function S_0 is locally semiconcave, i.e., for each compact, convex set $C \subset \mathbb{R}^n$ there exists $\alpha > 0$ such that $S_0(x) - \alpha |x|^2/2$ is a concave function of $x \in C$. Moreover, $S_0(x) \geq -K(1 + |x|)$ for some constant $K \geq 0$.

In generic terms, $\nabla^2 f$ signifies the Hessian matrix of a function $f \in C^2(\mathbb{R}^n)$. The notation $\nabla^2 f > 0$ means that $\nabla^2 f(p)$ is a positive definite matrix for every $p \in \mathbb{R}^n$. Condition (A) ensures that $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$ and $\nabla_p^2 H > 0$ in $\mathbb{R}^n \times \mathbb{R}^n$. We consider the functional

$$J^t(\boldsymbol{x}) = S_0(\boldsymbol{x}(0)) + \int_0^t L(\boldsymbol{x}(s), \dot{\boldsymbol{x}}(s)) \mathrm{d}s, \quad \boldsymbol{x} \in \mathcal{A}(t, x),$$

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