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Duffing equation and action functional

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ABSTRACT

In this paper, we investigate the periodic nonlinear second order ordinary differential equation with friction

$$x''(t) + cx'(t) + g(t, x(t)) = f(t), \quad t \in [0, T],$$
$$x(0) = x(T), \qquad x'(0) = x'(T),$$

where $c \in \mathbb{R}$, g is a Caratheodory function, $f \in L^1(0, T)$, a quotient $\frac{g(t,s)}{s}$ lies between 0 and $\frac{c^2}{4} + (\frac{2\pi}{T})^2$ and a nonlinearity g satisfies the potential Landesman–Lazer type condition. We introduce a corresponding energy functional and prove that its critical point is a classical solution to this problem.

1. Introduction

In this article, we study the nonlinear periodic boundary value problem

$$\begin{aligned} &x''(t) + cx'(t) + g(t, x(t)) = f(t), \quad t \in [0, T], \\ &x(0) = x(T), \qquad x'(0) = x'(T), \end{aligned} \tag{1.1}$$

where $c \in \mathbb{R}$, the nonlinearity $g: [0, T] \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and $f \in L^1(0, T)$.

In papers [1,2] the authors used topological degree arguments and supposed that $\gamma(t) \le \liminf_{|s|\to\infty} \frac{g(t,s)}{s} \le \limsup_{|s|\to\infty} \frac{g(t,s)}{s} \le \Gamma(t)$, $\Gamma(t) \le \left(\frac{2\pi}{T}\right)^2$ with the strict inequality on a subset of $[0, T]$ of positive measure (i.e. $\Gamma(t) \ll \left(\frac{2\pi}{T}\right)^2$) and $\gamma(t)$ satisfies $\int_0^T \gamma(t)\,\mathrm{d}t \ge 0$, $\int_0^T \gamma^+(t)\,\mathrm{d}t > 0$ where $\gamma^+(t) = \max_{t\in[0,T]}\{\gamma(t), 0\}$. In paper [3] the authors assume that, $\beta(t) \ll \frac{g(t,s)}{s} \ll \frac{c^2}{4} + \left(\frac{2\pi}{T}\right)^2$ for all $s \in \mathbb{R}$, where $\beta(t) \in C([0, T])$, $\beta(0) = \beta(T)$ and $\frac{1}{T}\int_0^T \beta(t)\,\mathrm{d}t > 0$.

In this article, we choose another strategy of proof which relies essentially on a variational method (see also [4,5]). We will investigate the functional

$$J(x) = \frac{1}{2}\int_0^T \left[\mathrm{e}^{ct}(x' + cx)^2\right]\mathrm{d}t - \int_0^T \left[\mathrm{e}^{ct}G(t, x) - \mathrm{e}^{ct}fx\right]\mathrm{d}t$$

where $G(t, s) = \int_0^s g(t, \xi)\mathrm{d}\xi$, and prove that a critical point x_0 of the functional J is a solution to problem (1.1). We will assume that the nonlinearity g satisfies, $0 \le \liminf_{|s|\to\infty} \frac{g(t,s)}{s} \le \limsup_{|s|\to\infty} \frac{g(t,s)}{s} \ll \frac{c^2}{4} + \left(\frac{2\pi}{T}\right)^2$ and a potential Landesman–Lazer type condition $\int_0^T G_-(t)\mathrm{d}t < \int_0^T f(t)\mathrm{d}t < \int_0^T G_+(t)\mathrm{d}t$, where $G_+(t) = \liminf_{s\to+\infty} \frac{G(t,s)}{s}$, $g_-(t) = \limsup_{s\to-\infty} \frac{G(t,s)}{s}$.

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