Nonlinear Analysis 74 (2011) 6728–6732

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

A fixed point approach to stability of functional equations

Janusz Brzdęk [a], Jacek Chudziak [b], Zsolt Páles [c,*]

[a] Department of Mathematics, Pedagogical University, Podchorążych 2, PL-30-084 Kraków, Poland
[b] Department of Mathematics, University of Rzeszów, Rejtana 16 C, 35-959 Rzeszów, Poland
[c] Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary

ARTICLE INFO

Article history:
Received 9 December 2010
Accepted 30 June 2011
Communicated by Ravi Agarwal

Keywords:
Hyers–Ulam stability
Fixed point
Functional equation in single variable
Nonlinear operator

ABSTRACT

We prove a simple fixed point theorem for some (not necessarily linear) operators and derive from it several quite general results on the stability of a very wide class of functional equations in single variable.

Interest in the investigation of the problem of the stability of functional equations (now named the Hyers–Ulam stability) has been stimulated by a question of Ulam (cf. [1,2]) and several papers that appeared afterwards and had been motivated by it (see, e.g., [3,4,1,5]). However the first known result of this kind is due to Pólya and Szegö [6]. For more details and further references concerning that subject, we refer to [7–15].

A fixed point approach has already been applied in the investigation of Hyers–Ulam stability, e.g., in [16–20]. In this note we present a modification of it that seems to be more natural for the field of functional equations. Thus we obtain in particular several significant supplements and/or generalizations of already known results proved in [7,21–30] and concerning stability of functional equations in single variable. For more information on such functional equations we refer to [31,32].

Denote by $\mathbb{N}_+$ and $\mathbb{R}_+$ the sets of nonnegative integers and nonnegative real numbers, respectively. For our subsequent results, we take the following four hypotheses.

(H1) X is a nonempty set and (Y, d) is a complete metric space.
(H2) $f_1, \dots, f_k : X \to X$ and $L_1, \dots, L_k : X \to \mathbb{R}_+$ are given maps.
(H3) $\mathscr{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$d\big((\mathscr{T}\xi)(x), (\mathscr{T}\mu)(x)\big) \le \sum_{i=1}^{k} L_i(x) d\big(\xi(f_i(x)), \mu(f_i(x))\big), \quad \xi,\ \mu \in Y^X,\ x \in X. \tag{1}$$

(H4) Λ is a linear operator defined by

$$(\Lambda\delta)(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)) \tag{2}$$

for $\delta : X \to \mathbb{R}_+$ and $x \in X$. Obviously, Λ is monotone with respect to the pointwise ordering in $\mathbb{R}_+^X$ (provided that L_i is nonnegative).

* Corresponding author.
E-mail addresses: jbrzdek@ap.krakow.pl (J. Brzdęk), chudziak@univ.rzeszow.pl (J. Chudziak), pales@math.klte.hu (Z. Páles).

0362-546X/$ – see front matter
doi:10.1016/j.na.2011.06.052