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Gradient estimates and Liouville theorems for nonlinear parabolic equations on noncompact Riemannian manifolds

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ABSTRACT

Let M be a complete noncompact Riemannian manifold. In this paper, we derive a local gradient estimate for positive solutions of the nonlinear parabolic equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) + \lambda(x,t) u^{\alpha}(x,t) = 0$$

on $M \times (-\infty, 0]$. We also obtain a theorem of Liouville type for positive solutions of this nonlinear equation. This paper extends the result of Souplet and Zhang [P. Souplet, Qi S. Zhang, Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds, Bull. Lond. Math. Soc. 38 (2006) 1045–1053].

1. Introduction

Since the year of 1975, there have been plenty of results obtained concerning gradient estimates, the most outstanding of which were due to Cheng and Yau [1] and Li and Yau [2]. Gradient estimates have played an important role in the study of PDE, especially the Laplace equation and the heat equation. We do not want to state the results for them here. In this paper, we are concerned with Hamilton's gradient estimate (cf. [3]), which is more interesting in view of only having a spatial gradient in the term to be estimated. The wonderful thing is that, in the case of compact manifolds, Hamilton's estimate shows that one can compare the temperature of two different points at the same time provided that the temperature is bounded.

About four years ago, Souplet and Zhang [4] found a useful auxiliary function, after using the cut-off function of Li and Yau [2]; they obtained the following results:

Theorem A (*Souplet–Zhang [4]*). *Let* $\mathbf{M}$ *be a Riemannian manifold of dimension* $n \geq 2$ *with Ricci*$(\mathbf{M}) \geq -k$ *for some* $k \geq 0$. *Suppose that* u *is any positive solution to the heat equation in* $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset \mathbf{M} \times (-\infty, \infty)$. *Suppose also that* $u \leq M$ *in* $Q_{R,T}$. *Then there exists a dimensional constant* c *such that*

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leq c \left(\frac{1}{R} + \frac{1}{T^{\frac{1}{2}}} + \sqrt{k}\right)\left(1 + \log \frac{M}{u(x,t)}\right)$$

in $Q_{\frac{R}{2},\frac{T}{2}}$.

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