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Functional equations with exotic addition

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ABSTRACT

We deal with the following "exotic" addition:

$$x \oplus y := xf(y) + yf(x)$$

on an arbitrary interval containing zero and we solve "exotic" versions of the Jensen functional equation and the equation of derivations.

1. Introduction

Northshield [1] introduced the term exotic addition for two types of operations he dealt with. One of them was the following "sine-type" addition on the real line:

$$x \oplus y := xf(y) + yf(x) \tag{1}$$

with $f: \mathbb{R} \to \mathbb{R}$, enjoying some regularity properties. The name "sine-type" addition, which appears in [1], is justified by the fact that if one takes mapping $f: [-1, 1] \to \mathbb{R}$ given by $f(x) = \sqrt{1 - x^2}$ for $x \in [-1, 1]$, then in this particular case the sine function acts as a homomorphism between the real line with ordinary addition and the interval $[-1, 1]$ with "exotic" addition $\oplus$. In other words, we have

$$\sin(x + y) = \sin(x) \oplus \sin(y)$$

for all $x, y \in \mathbb{R}$.

Northshield [1] provided several conditions equivalent to the associativity of $\oplus$ [1, Theorem 4 and Corollary 1]. From these results, it follows that the associativity seems to be a fairly restrictive assumption since it implies a particular form of mapping f (given in an implicit form involving solutions of some ODEs).

It is easy to check that the associativity of a similar operation $\odot$ defined as $x \odot y = x + f(x)y$ for all x, y belonging to some nonvoid interval is equivalent to the fact that function f solves the Gołąb–Schinzel functional equation:

$$f(x + yf(x)) = f(x)f(y), \tag{2}$$

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