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Retracting a ball onto a sphere in some Banach spaces

Łukasz Piasecki

Institute of Mathematics, Maria Curie-Sklodowska University, 20-031 Lublin, Poland

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ABSTRACT

The unit sphere in an infinite dimensional Banach space is a lipschitzian retract of the unit ball. The aim of this paper is to present a new upper bound for the optimal retraction constant in some classical Banach spaces. In particular, an improved estimate from above is obtained for the space $C[0, 1]$.

1. Introduction

Let $(X, \|\ \|)$ be an infinite dimensional real Banach space with the closed unit ball B_X and the unit sphere S_X. Since the works of Nowak [1] and Benyamini and Sternfeld [2] it has been known that S_X is a lipschitzian retract of B_X. This means that there exists a retraction $R : B_X \to S_X$ ($R|_{S_X} = I$ where I denotes identity) satisfying, with a certain constant $k > 0$, the Lipschitz condition

$$\|Rx - Ry\| \leq k\, \|x - y\| \tag{1.1}$$

for all $x, y \in B_X$. For any space X we define the *optimal retraction constant*

$$k_0(X) = \inf\{k :\ \text{there exists a retraction } R : B_X \to S_X \text{ satisfying (1.1)}\}.$$

At present the exact value of $k_0(X)$ is not known for any single Banach space. Let us recall some up to date bounds. The universal known bound from below is $k_0(X) \geq 3$ but for some spaces there are better estimates, for example, $k_0(X) > 3$ for uniformly convex spaces, $k_0(l_1) \geq 4$ and $k_0(H) > 4.5$ for Hilbert space. There have been several approaches to giving a reasonable universal estimate from above. All of them ended at the level of the high thousands (see for example [3]). Much better estimates exist for many classical Banach spaces; for example $k_0(H) \leq 28.99$ for every Hilbert space (see [4]). A construction presented in [5] shows that $k_0(l_1) \leq 8$. The same estimate holds for $L_1(0, 1)$ and a few other spaces (see [6]). In paper [7] the authors present an example of 7-lipschitzian retraction in the space $BC_z(M)$, where M is an arbitrary connected metric space consisting of more than one point, $z \in M$ and $BC_z(M)$ is the space of bounded continuous functions vanishing at z and furnished with a standard sup norm. A more technical approach is presented in [8] with the conclusion $k_0(BC_z(M)) \leq 2(2 + \sqrt{2}) = 6.82\cdots$. At present the last result is the minimum of upper bounds over the Banach spaces for which the upper bound is known.

An interesting situation is observed for the space $C[0, 1]$. The best known estimate for this space is $k_0(C[0, 1]) \leq 4(1 + \sqrt{2})^2 = 23.31\cdots$ (see [9]). Here we describe a new general construction improving, in particular, the upper bound of $k_0(C[0, 1])$.

E-mail address: piasecki@hektor.umcs.lublin.pl.

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