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Analytic description and explicit parametrisation of the equilibrium shapes of elastic rings and tubes under uniform hydrostatic pressure

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ABSTRACT

The parametric equations of the plane curves determining the equilibrium shapes that a uniform inextensible elastic ring or tube could take subject to a uniform hydrostatic pressure are presented in an explicit analytic form. The determination of the equilibrium shape of such a structure corresponding to a given pressure is reduced to the solution of two transcendental equations. The shapes with points of contact and the corresponding (contact) pressures are determined by the solutions of three transcendental equations. The analytic results presented here confirm many of the previous numerical results on this subject but the results concerning the shapes with lines of contact reported up to now are revised.

1. Introduction

In the present paper, the problem for determination of the equilibrium shapes of a circular inextensible elastic ring subject to a uniformly distributed external force that acts normally to the ring in the ring plane is addressed. This problem is also referred to as the stability problem or buckling of the circular shape of the ring and the other equilibrium shapes are called buckled [1–3]. It is also known (see, e.g., [4–6]) that if a cylindrical elastic shell of circular cross section (i.e., a tube) is subject to a uniform external pressure, which is normal to its middle surface, then the typical cross section of the deformed tube takes the same shapes as the axis of a deformed elastic ring does provided that the latter is a simple curve (i.e., a curve without intersections). Therefore, here the term "ring" will be used to indicate both a proper ring and a tube. It should be noted also that in the majority of the works in this field, the distributed force acting on a ring is called pressure as in the case of a shell. Following this tradition, we will use the same term in the present study remembering that pressure means force per unit length in the case of a ring and force per unit area in the case of a shell.

Maurice Lévy [7] was the first who stated and studied the problem under consideration and reduced the determination of the foregoing equilibrium shapes in polar coordinates to two elliptic integrals for the arclength and polar angle regarded as functions of the squared radial coordinate. He found also several remarkable properties of the equilibrium ring shapes and obtained that if the pressure p is such that $p<(9/4)(D/\rho^3)$, where D and ρ are the ring bending rigidity and radius of the undeformed shape, respectively, then the ring possesses only the circular equilibrium shape.

Later on, Halphen [8] and Greenhill [9] derived exact solutions to this problem in terms of Weierstrass elliptic functions on the ground of complicated analyses of the properties of the aforementioned elliptic integrals. Halphen (see [8, p. 235]) found out that non-circular shapes with $n \geq 2$ axes of symmetry are possible only for pressures greater than $p_n=(n^2-1)(D/\rho^3)$. Halphen [8] and Greenhill [9] presented also several examples of non-circular equilibrium ring shapes. It should be noted, however, that the exact solutions reported in [8,9], representing the polar angle as a function of the radius, appeared to be intractable and many researchers continued searching exact solutions [1,10–15], while others used various approximations [2,4,6,16] on the way to determine the equilibrium shapes of the ring.

Carrier [1] was the first who reconsidered the foregoing problem for the buckling of an elastic ring about half a century after the works by Lévy, Halphen and Greenhill. He expressed the curvature of the deformed ring in terms of Jacobi cosine function [17] involving several unknown parameters to be determined by a system of algebraic equations. However, he succeeded to find approximate solutions to this system only for small deflections from the undeformed circular ring shape (see the exhaustive analysis provided recently by Adams [11] who has criticised and developed Carrier's work [1]).

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