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Mathematics and Computers in Simulation 82 (2012) 1030–1037

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

Original article

Semiclassical spectral confinement for the sine-Gordon equation

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Received 16 October 2009; received in revised form 14 May 2010; accepted 1 July 2010
Available online 14 July 2010

Abstract

The inverse scattering method for solving the sine-Gordon equation in laboratory coordinates requires the analysis of the Faddeev–Takhtajan eigenvalue problem. This problem is not self-adjoint and the eigenvalues may lie anywhere in the complex plane, so it is of interest to determine conditions on the initial data that restrict where the eigenvalues can be. We establish bounds on the eigenvalues for a broad class of zero-charge initial data that are applicable in the semiclassical or zero-dispersion limit. It is shown that no point off the coordinate axes or turning point curve can be an eigenvalue if the dispersion parameter is sufficiently small.

Sine-Gordon; Semiclassical problems; Eigenvalues; Inverse scattering

1. Introduction

The sine-Gordon equation in laboratory coordinates

$$\varepsilon^2 u_{tt} - \varepsilon^2 u_{xx} + \sin(u) = 0 \tag{1}$$

is a basic model for a variety of physical systems. When used to model idealized magnetic flux propagation along the insulating barrier between two superconductors in a Josephson junction, it is appropriate to consider the semiclassical or zero-dispersion limit as $\varepsilon \downarrow 0$ [9]. Specifically, we fix two continuous functions $f(x) : \mathbb{R} \to \mathbb{R}$ and $g(x) : \mathbb{R} \to \mathbb{R}$, independent of ε, and pose the Cauchy problem for (1) with initial data

$$u(x, 0; \varepsilon) = f(x), \quad \varepsilon u_t(x, 0; \varepsilon) = g(x). \tag{2}$$

For physical applications, one can assume

$$\lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to+\infty} f(x) = 2\pi n \text{ for } n \in \mathbb{Z}, \quad \lim_{|x|\to+\infty} g(x) = 0. \tag{3}$$

The number n, which is a constant of motion, is the *winding number* or *topological charge* of the solution. We will be considering the zero-winding or zero-charge case of $n = 0$.

The sine-Gordon equation is solvable by the inverse-scattering method. Each (reasonably well-behaved) set of Cauchy data characterizes a set of reflection data, consisting of eigenvalues, modified proportionality constants, and the reflection coefficient. Finding the scattering data from the Cauchy data comprises the *forward scattering transform*. The scattering data can then be evolved forward in time, and then in principle the solution to (1) at a later time can

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doi:10.1016/j.matcom.2010.07.002