A nonstandard finite difference scheme for advection diffusion equation

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Abstract

In this paper, numerical solution of the advection diffusion equation is presented based on the nonstandard finite difference (NSFD) scheme. At first two exact finite difference schemes for the advection equation are obtained. Afterwards, an NSFD scheme is presented for this equation. The stability of this scheme is discussed. Then, the NSFD scheme for the advection equation is combined with a finite difference approximation to provide an NSFD scheme for advection diffusion equation. This new scheme could prserve the positivity and boundedness of the solution of the original equation. The numerical results obtained by the NSFD scheme is compared with the exact solution and some available methods to verify the accuracy and efficiency of the NSFD scheme.

1 Introduction

Nonlinear partial differential equations (PDEs) play an important role in the various fields of physical science and engineering. One of the important nonlinear PDE which appears in various applications, is the advection diffusion equation. In this paper we consider the advection diffusion of the form

$$u_t + \beta_1 u_x = \beta_2 u_{xx} + \lambda u - \lambda u^\alpha, \tag{1}$$

where the coefficients β_1, β_2, λ and α are nonnegative constants. NSFD schemes which introduced by Mickens to solve ordinary differential equations (ODEs) and PDEs [2, 3], have been proved to be one of the most efficient methods in recent years. Instead of classical methods, NSFD schemes can alternatively be used to obtain more qualitative results and remove numerical instabilities. These schemes are developed for compensating the weaknesses, such as numerical instabilities that may be caused by standard finite difference (SFD) methods [4, 5, 6, 7, 8, 9, 10]. In this paper, we try to construct an NSFD scheme that could preserve the positivity and boundedness of the solution of (1).

2 Analysis of exact finite difference schemes

In this section, we obtain two exact finite difference schemes for the following advection equation

$$u_t + \beta_1 u_x = \lambda u - \lambda u^\alpha. \tag{2}$$

Along the direction of characteristics [1], equation (2) can be translated into a group of ODEs as follows

$$\begin{aligned} x'(t) &= \beta_1, \\ w'(t) &= \lambda w(t) - \lambda w^\alpha(t), \\ x(t_0) &= x_0, \\ w(t_0) &= h(x_0), \end{aligned}$$

where $w(t) = u(x(t), t)$ and $x(t) = \beta_1(t - t_0) + x_0$ denotes a characteristic of (2). For $w(t_0) > 0$ and $\alpha > 1$, the analytical solution

$$w'(t) = \lambda w(t) - \lambda w^\alpha(t), \tag{3}$$

is

$$w(t) = [e^{(1-\alpha)\lambda(t-t_0)}((w^{1-\alpha}(t_0) - 1) - 1)]^{\frac{1}{1-\alpha}}.$$

Let Δx and Δt denote the space and time step sizes, respectively. Set $x_j = j\Delta x$ for $j \in Z$ and $t_n = n\Delta t$ for $n \in N$. Define $w_{n+1} = w(t_{n+1})$ and $w_n = w(t_n)$.