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A Reparameterisation Based Approach to Geodesic Constrained Solvers for Curve Matching

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Abstract We present a numerical algorithm for a new matching approach for parameterisation independent diffeomorphic registration of curves in the plane, targeted at robust registration between curves that require large deformations. This condition is particularly useful for the geodesic constrained approach in which the matching functional is minimised subject to the constraint that the evolving diffeomorphism satisfies the Hamiltonian equations of motion; this means that each iteration of the nonlinear optimisation algorithm produces a geodesic (up to numerical discretisation). We ensure that the computed solutions correspond to geodesics in the shape space by enforcing the horizontality condition (conjugate momentum is normal to the curve). Explicitly introducing and solving for a reparameterisation variable allows the use of a point-to-point matching condition. The equations are discretised using the variational particle-mesh method. We provide comprehensive numerical convergence tests and benchmark the algorithm in the context of large deformations, to show that it is a viable, efficient and accurate method for obtaining geodesics between curves.

Keywords Geodesic shooting · Computational anatomy · Curve registration

1 Introduction

Computational anatomy, the computational study of shape variability in anatomical structures, has been a very active area of study for the last fifteen years (Grenander and Miller 1998; Miller and Younes 2001). The fundamental principle of computational anatomy is that shapes of curves, surfaces, scalar images, diffusion tensor images and other objects may be studied in terms of the relative deformations that take one shape to another. To prevent overlapping and cavitation of these objects, the deformations are described as fluid flows in which the shape is deformed by advection by a time-dependent velocity field over a fixed time interval. In general, there are infinitely many of such velocity fields; to uniquely characterise the flow-induced deformation that takes one shape to another, the time-dependent velocity field is chosen that minimises a given energy that includes terms that penalise high derivatives in the velocity field (such as a Sobolev norm of the velocity field). This defines the distance along a path between two shapes, and hence the shortest path, or geodesic. This allows shapes to be described relative to a template shape using the initial direction of the geodesic evolution; this direction is characterised by a conjugate momentum variable. Since the momentum is defined on a vector space, linear statistical techniques such as wavelet analysis, sparse compression or Markov Chain Monte-Carlo may be performed. This makes Computational Anatomy a very powerful tool for quantifying the difference between shapes and making statistical inferences about them. One direction where the approach has proved very successful is in the generalisation of Principal Component Analysis to this type of shape analysis (Fletcher et al. 2004, 2008; Fletcher 2004). As argued in Vialard et al. (2012), the utility of momentum for statistical analysis motivates solving using a geodesic constrained, or Hamiltonian, approach, constraining the optimisation algorithm so that a reproducable numerical solution of the geodesic equations is obtained at each stage. In this approach (also advocated in McLachlan and Marsland 2007; Younes 2007; Cotter 2008),

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