International Journal of Solids and Structures 48 (2011) 2496–2510

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Couple stress theory for solids

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ARTICLE INFO

Article history:
Received 8 September 2010
Received in revised form 19 March 2011
Available online 7 May 2011

Keywords:
Couple stress
Size-dependent solid mechanics
Elasticity
Curvature tensor
Micro- and nano-mechanics
Size effect

ABSTRACT

By relying on the definition of admissible boundary conditions, the principle of virtual work and some kinematical considerations, we establish the skew-symmetric character of the couple-stress tensor in size-dependent continuum representations of matter. This fundamental result, which is independent of the material behavior, resolves all difficulties in developing a consistent couple stress theory. We then develop the corresponding size-dependent theory of small deformations in elastic bodies, including the energy and constitutive relations, displacement formulations, the uniqueness theorem for the corresponding boundary value problem and the reciprocal theorem for linear elasticity theory. Next, we consider the more restrictive case of isotropic materials and present general solutions for two-dimensional problems based on stress functions and for problems of anti-plane deformation. Finally, we examine several boundary value problems within this consistent size-dependent theory of elasticity.

1. Introduction

Classical continuum mechanics is an approximation based on the assumption that matter is continuously distributed throughout the body. This theory provides a reasonable basis for analyzing the behavior of materials at the macro-scale, where the microstructure size-dependency can be neglected. Experiments show, however, that the mechanical behavior of materials in small scales is different from their behavior at macro-scales. Any attempt to drop the continuity assumption in a modified theory is bound to make the analysis extremely difficult and computationally intensive. Therefore, we need to develop a consistent size-dependent continuum mechanics, which accounts for the microstructure of materials. This theory must span many scales and, of course, reduce to classical continuum mechanics for macro-scale size problems.

New measures of deformation, which are length related, such as the curvature tensor, are needed in a more complete continuum theory. As a consequence, such a theory will also require the introduction of couple-stresses. The existence of couple-stress in materials was originally postulated by Voigt (1887). However, Cosserat and Cosserat (1909) were the first to develop a mathematical model to analyze materials with couple-stresses. In the original Cosserat theory, the kinematical quantities were the displacement and a material microrotation, hypothesized to be independent of the continuum mechanical rotation. This latter quantity, which may be called the macrorotation, is the usual rotation vector defined as one half of the curl of the displacement field.

A couple stress theory, using macrorotation as the true kinematical rotation, was developed much later by Toupin (1962), Mindlin and Tiersten (1962), Koiter (1964), and others for elastic bodies. In these developments, the gradient of the rotation vector is used as a curvature tensor. Unfortunately, there are some difficulties with these formulations. Perhaps the most disturbing troubles are the indeterminacy of the spherical part of the couple-stress tensor and the appearance of the body couple in the constitutive relation for the force-stress tensor (Mindlin and Tiersten, 1962). This inconsistent theory is called the indeterminate couple stress theory in the literature (Eringen, 1968). As a result of the inconsistency, a number of alternative theories have been developed.

One branch revives the idea of microrotation, inherited from Cosserat and Cosserat (1909) and is called micropolar theories (e.g., Mindlin, 1964; Eringen, 1968; Nowacki, 1986; Chen and Wang, 2001). However, microrotation, which brings extraneous degrees of freedom, is not a proper continuum mechanical concept. How can the effect of the discontinuous microstructure of matter be represented mathematically by an artificial continuous microrotation? Thus, a consistent size-dependent continuum mechanics theory should involve only true continuum kinematical quantities without recourse to any additional artificial degrees of freedom.

The other main branch, labeled second gradient theories, avoids the idea of microrotation by introducing gradients of strain, rotation or various combinations thereof (e.g., Mindlin and Eshel, 1968; Yang et al., 2002; Lazar et al., 2005). Although these theories use true continuum representations of deformation, the resulting formulations are not consistent with proper boundary condition specifications and energy conjugacy within the principle of virtual work.

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doi:10.1016/j.ijsolstr.2011.05.002