Constructing an $\mathcal{H}$-matrix via Randomized Algorithms

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Abstract

The key point in constructing an $\mathcal{H}$-matrix is to approximate certain subblocks $\boldsymbol{D}_{n'\times m'}$ of a dense matrix $\boldsymbol{A}_{n\times m}$ by data-sparse low-rank matrices that can be represented as $\boldsymbol{R}_{n'\times m'} = \boldsymbol{U}_{n'\times k} \cdot \boldsymbol{V}^T_{k\times m'}$, with $k \ll \min\{n', m'\}$ as the actual rank of $\boldsymbol{R}$. To obtain $\boldsymbol{R}$ from $\boldsymbol{D}$, the most accurate method is based on SVD which is computationally expensive and needs $\mathcal{O}(n'm' \min\{n', m'\})$ operations. In this paper, we consider various randomized algorithms to obtain such approximations with cost $\mathcal{O}(m'n'k)$. We confirm the advantages of these algorithms applied to a BEM model numerically.

Keywords: Hierarchical matrices, low-rank approximation, randomized algorithm
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1 Introduction

$\mathcal{H}$-matrices provide an inexpensive but sufficiently accurate approximation to dense matrices as they appear in boundary element methods (BEM). Solving integral equations by BEM, finally lead to a linear system of equations:

$$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}. \tag{1}$$

The resulting matrix $\boldsymbol{A}_{n\times n}$ is dense and requires complexity $\mathcal{O}(n^2)$ for its storage as well as matrix-vector multiplication. For computing matrix-matrix multiplication and inversion, this cost would be $\mathcal{O}(n^3)$, which for large-scale computations is prohibitively expensive. The hierarchical matrix technique provides a data-sparse structure by which all $\mathcal{H}$-matrix arithmetic can be performed in almost optimal complexity $\mathcal{O}(n \log^q n)$ with moderate constant q.

To build an $\mathcal{H}$-matrix approximation $\boldsymbol{A}_{\mathcal{H}}$ to a given dense matrix $\boldsymbol{A}$, a tree like data-sparse structure is used to store $\boldsymbol{A}$ such that the leaves of the tree are dense or low-rank matrices ($\mathcal{R}(k)$-matrices). A low-rank matrix stored in so-called $\mathcal{R}(k)$-format in the following sense:

Definition 1.1. A matrix block $\boldsymbol{R}_{n'\times m'}$, is called to be stored in an $\mathcal{R}(k)$-matrix representation, if we have $\boldsymbol{R} = \boldsymbol{U} \cdot \boldsymbol{V}^T$, where the two matrices $\boldsymbol{U}_{n'\times k}$ and $\boldsymbol{V}_{m'\times k}$ are dense matrices. We call $\boldsymbol{R}$ a low-rank or $\mathcal{R}(k)$-matrix.

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