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Analysis of nonsmooth vector-valued functions associated with infinite-dimensional second-order cones

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ABSTRACT

Given a Hilbert space $\mathcal{H}$, the infinite-dimensional Lorentz/second-order cone $\mathbb{K}$ is introduced. For any $x \in \mathcal{H}$, a spectral decomposition is introduced, and for any function $f : \mathbb{R} \to \mathbb{R}$, we define a corresponding vector-valued function $f^{\mathcal{H}}(x)$ on Hilbert space $\mathcal{H}$ by applying f to the spectral values of the spectral decomposition of $x \in \mathcal{H}$ with respect to $\mathbb{K}$. We show that this vector-valued function inherits from f the properties of continuity, Lipschitz continuity, differentiability, smoothness, as well as s-semismoothness. These results can be helpful for designing and analyzing solution methods for solving infinite-dimensional second-order cone programs and complementarity problems.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle$, and we write the norm induced by $\langle\cdot,\cdot\rangle$ as $\|\cdot\|$. For any given closed convex cone $K \subseteq \mathcal{H}$,

$$K^* := \{x \in \mathcal{H} \mid \langle x, y\rangle \geq 0,\ \forall y \in K\}$$

is the dual cone of K. A closed convex cone K in $\mathcal{H}$ is called *self-dual* if K coincides with its dual cone K^*; for example, the non-negative orthant cone $\mathbb{R}^n_+$ and the second-order cone (also called Lorentz cone) $\mathbb{K}^n := \{(r, x') \in \mathbb{R} \times \mathbb{R}^{n-1} \mid r \geq \|x'\|\}$. As discussed in [1], this Lorentz cone $\mathbb{K}^n$ can be rewritten as

$$\mathbb{K}^n := \left\{x \in \mathbb{R}^n \mid \langle x, e\rangle \geq \frac{1}{\sqrt{2}}\|x\|\right\} \quad \text{with } e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

This motivates us to consider the following closed convex cone in the Hilbert space $\mathcal{H}$:

$$K(e, r) := \{x \in \mathcal{H} \mid \langle x, e\rangle \geq r\|x\|\}$$

where $e \in \mathcal{H}$ with $\|e\| = 1$ and $r \in \mathbb{R}$ with $0 < r < 1$. It can be seen that $K(e, r)$ is pointed, i.e., $K(e, r) \cap (-K(e, r)) = \{0\}$. Moreover, by denoting

$$\langle e\rangle^{\perp} := \{x \in \mathcal{H} \mid \langle x, e\rangle = 0\},$$

we may express the closed convex cone $K(e, r)$ as

$$K(e, r) = \left\{x' + \lambda e \in \mathcal{H} \mid x' \in \langle e\rangle^{\perp} \text{ and } \lambda \geq \frac{r}{\sqrt{1 - r^2}}\|x'\|\right\}.$$

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