



# Some properties of two-phase quadrature domains

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## ARTICLE INFO

### Article history:

Received 15 January 2011

Accepted 8 February 2011

### MSC:

primary 35R35  
35B06

### Keywords:

Quadrature domain  
Two-phase free boundary problems  
Uniqueness

## ABSTRACT

In this paper, we investigate general properties of the two-phase quadrature domain, which was recently introduced by Emamizadeh, Prajapat and Shahgholian. The concept, which is a generalization of the well-known one-phase domain, introduces substantial difficulties with interesting features even richer than those of the one-phase counterpart.

For given positive constants  $\lambda^\pm$  and two bounded and compactly supported measures  $\mu^\pm$ , we investigate the uniqueness of the solution of the following free boundary problem:

$$\begin{cases} \Delta u = (\lambda^+ \chi_{\Omega^+} - \mu^+) - (\lambda^- \chi_{\Omega^-} - \mu^-), & \text{in } \mathbb{R}^N \ (N \geq 2), \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1)$$

where  $\Omega = \Omega^+ \cup \Omega^-$ . It is further required that the supports of  $\mu^\pm$  should be inside  $\Omega^\pm$ ; this in general may fail and give rise to non-existence of solutions.

Along the paths to various properties that we state and prove here, we also present several conjectures and open problems that we believe should be true.

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## 1. Introduction

The concept of quadrature domains is well known in modern potential theory and concerns the generalized form of the (sub)mean value property for (sub)harmonic functions.

The main idea in this paper is to deal with a two-phase version of this concept, introduced in [1]. Our main result concerns uniqueness for two-phase quadrature domains when certain restrictions are imposed on the sign(s) of the solution function.

This paper is organized as follows. Section 2 contains some background for the one-phase case and some fundamental concepts in potential theory. In Section 3 we then move to the two-phase case scenario and extract its PDE formulation, introduce quadrature inequalities and give some examples. In Section 4 we note some recent results on existence theory for two-phase free boundary problems and finally in the last section we study the uniqueness and prove our main result just by considering some conditions. Also we make some conjectures.

## 2. The one-phase case

The definition of a quadrature domain is as follows.

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